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BRAIDING

– STANDARD HERRINGBONE PINEAPPLE KNOTS

A Series of Books on Braiding

Book 4/1 * * * * * 1991

New and Automatic Construction Methods

A Series of Books on Braiding

Book 4/1

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BRAIDING

NEW AND AUTOMATIC METHODS FOR CONSTRUCTING KNOTS AND BRAIDS

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BOOK 4/1

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A SERIES OF BOOKS ON BRAIDING

1991

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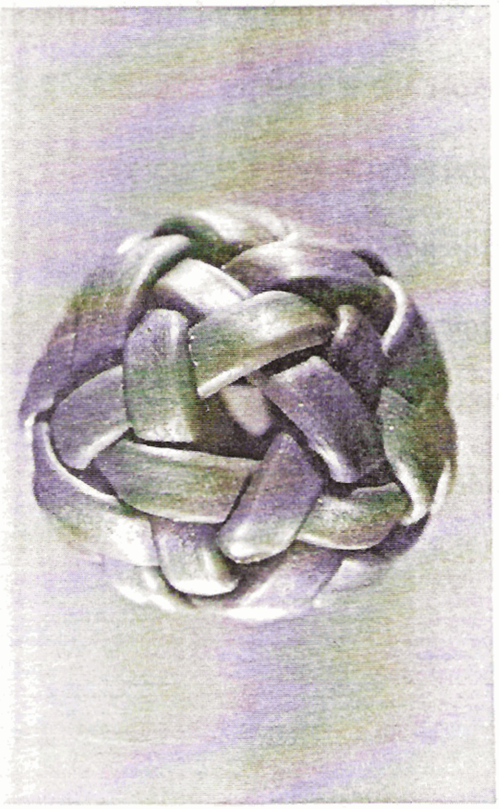


Some bookes are to bee tasted,
others to bee swallowed,
and some few to bee chewed and disgested.
— FRANCIS BACON, *Essayes* (1597)

FOREWORD

This book is the third of a series whose aim is to present new theories and methods of braiding. Each book treats one or more types of braid, and gives methods and formulae which enable artisans to produce similar braids of their own design. The methods are based upon mathematical principles discovered by Georg Schaake in the early '80s. These principles are based in the main on the operations of **modular** (or **clock**) arithmetic. For those not familiar with these operations we include a brief introduction to the topic of modular arithmetic in Appendix 3.

This volume deals with the family of The Standard Herringbone Pineapple Knots. It presents tabular methods which enable the reader to braid these knots with a minimum of mathematical knowledge. As with all books in this Series, the underlying mathematical theory is not discussed in detail. This theory is presented in an associated research report.
Further, a computer diskette will be made available for computing automatically the braiding algorithms for these Pineapple Knots, for those artisans who do not wish to use the tables given in this book.

Details of research reports for this and other books in the series, and of the computer diskettes currently available, may be had by writing to Dr. J.C.Turner, Department of Mathematics and Statistics, University of Waikato, Hamilton, New Zealand.

We shall not give practical details on braid production, such as what kinds of tools and thongs to use. Readers requiring this kind of information are referred to the excellent encyclopaedia of rawhide and leather braiding written by Bruce Grant (see the Bibliography for details and other references).

The authors will be pleased to receive comments from readers on the new methods given in the books. Suggestions for improving the manner of their presentation will also be appreciated.

As well as the books and the research reports, the authors are now producing a Series of Pamphlets entitled **Topics in the Theory and Practice of Braiding**. They present short introductions to various braiding topics, which are not necessarily directly linked with subjects dealt with in the Book-Series. The first three in the Pamphlet-Series are **(1) Introducing Grid-Diagrams in Braiding**; **(2) Edge Lacing — the Double Cordovan Stitch**; and **(3) Braiding Application — Horse Halter**. These publications may be obtained from the address given above.

A.G.S.
J.C.T.
D.A.S.

October, 1991

1

INTRODUCTION

Difficulties and obscurities will disappear
when we don't rush, and start at the beginning.

This volume in our series of books on braiding deals specifically with the **Standard Herringbone Pineapple Knots.** An example of these Knots is described by Bruce Grant on page 420 (plates 173 and 174) in his book *"Encyclopedia of Rawhide and Leather Braiding"*; Grant refers to it (on page 134) as being the "King of the braided Knots". These Knots form a Herringbone Pineapple Knot sub-family. Any member of this sub-family is a special interbraid of Turk's Head Knots, all having the same number of bights.

These interbraided Turk's Head Knots have an **odd** number of parts; they all have either the same odd number of parts, or else they form two sets: all the members of a set have the same odd number of parts, and this number of parts in one set differs from the number of parts in the other set by 2.

The Cylindrical Braid-class to which the Pineapple Knots belong is characterised by the regular **nesting** of the bights at the two parallel edges of the braid. This Braid-class we call **the Regular-Nested Cylindrical Braids.** It contains an enormous number of different knot-classes, each of which has its own specific features.

It is of interest to note that some braids which cover a sphere belong to the Regular-Nested Cylindrical Braids, but there are a great many spherical braids which do not belong to this Braid-class.

Although Bruce Grant clearly indicates in his writings the specific features of Pineapple Knots, and hence the Pineapple Knot-class, writers recently seem to call any knot belonging to the Regular-Nested Cylindrical Braid-class a Pineapple Knot. They then wonder why they are unable to find the relationships which govern these Knots. It should be obvious that we have to start at the beginning, and that is with the study of the **string-runs** (see Ref. [1]) of braids; this will lead to **braid classes**, each of which, in general, divides into **braid sub-classes**. Each of these braid sub-classes may in turn divide into **braid families**; and each of these may divide into **braid sub-families**.

Only after this classification has been completed can we start with the study of the consequences imposed by the different weaving patterns: the **coding** (see Ref. [1]). This study will lead to **knot classes**, which in general divide into **knot sub-classes**. Each of these knot sub-classes may in turn divide into **knot families**; and each of these may divide into **knot sub-families**.

Consequently, before we are able to explain clearly the various specific properties of the Standard Herringbone Pineapple Knots, we first have to describe the general nature of the Regular-Nested Cylindrical Braids; this is done in Chapter 2. In Chapter 3, we start by giving a definition of the **Pineapple Knot class** by means of the **essential coding** of its members. This Knot class consists of many sub-classes. One of these is the **Herringbone Pineapple Knot sub-class**, which is divided into two families: one the **Herringbone Pineapple Knot family**, and the other the **Broken-Herringbone Pineapple Knot family** (see Appendix 2). We show how the Herringbone Pineapple Knot family is arrived at, and briefly discuss some properties associated with its sub-families.

We have then reached the stage where a more detailed treatment of the **Standard Herringbone Pineapple Knots** can follow; this is carried out in Chapter 4.

Those braiders who are interested in learning only the required procedures for obtaining the braiding algorithms associated with the Standard Herringbone Pineapple Knots, can start with Chapter 5. If they do so, however, they will lack a substantial amount of insight, but nevertheless they will be able to braid any Standard Herringbone Pineapple Knot.

The necessary braiding algorithms for the construction of these Knots can be obtained in a simple manner from tables. The procedures involved are fully explained in Chapter 5. Readers who are devoid of any mathematical knowledge can dispense with the formulae on pages 36 and 37; they will have to establish the values of the parameters by trial and error, which is, in many cases, an easy matter. To help the reader in becoming thoroughly familiar with the procedures, two fully worked examples are given.

It is convenient to divide the Standard Herringbone Pineapple Knots into Types; this enables the braider to specify unambiguously a particular Standard Herringbone Pineapple Knot. This procedure is discussed in detail in Chapter 6.

To enable the construction of Algorithm-tables for values of B^* and s which are greater than the ones catered for in the given tables, we give in Chapter 7 the method to do so. It is closely related to the calculation method employed in obtaining the braiding algorithms for Regular Knots as described in our book *"Braiding — Regular Knots"* under method I on page 28 (see Ref. [1]). The relationship between the two methods is made clear in Appendix 4.

A summary of the classification of the Regular-Nested Cylindrical Braids is given in Appendix 1. Since braids can be braided in two ways, according as their cycles are laid down from lower-left to upper-left (upwards braiding), or from upper-left to lower-left (downwards braiding), this Appendix is accordingly divided into two sections, I and II.

Appendix 2 gives a summary of the Herringbone Pineapple Knot sub-class. This summary is also divided into two sections: Section I for upwards braiding, and Section II for downwards braiding.

The reader should note that some formulae depend on which of the two braiding directions is used.

It should be stressed that the Appendices 1 and 2 should not be studied before the reader is thoroughly conversant with the contents of the Chapters 2, 3 and 4. Only after the contents of these Chapters are fully understood and properly "digested", will the reader be able to follow the material in these Appendices without difficulty.

Since the calculation methods in braiding rely heavily on modular arithmetic, we have included a brief introduction to it in Appendix 3. It gives the reader all he needs to know about this natural form of arithmetic concerning braiding.

There are no shortcuts to the understanding of braiding processes, and those who think that they will be able to comprehend these processes by theory or practice alone are badly mistaken. It is essential that the reader acquires a large measure of practical experience, since it is a prerequisite for the successful understanding and development of the necessary theory. Braids are basically geometric objects, consequently grid-diagrams are one of the best means for representing them. It is therefore important that the reader draws up many grid-diagrams and executes the braiding process involved for each. Only then will he acquire the skill in "reading" them, so that they are automatically translated into the geometrical objects they represent, which we call braids or knots.

The Chapters that follow are to be chewed and digested thoroughly, if competence in designing and constructing these braids is to be achieved. We hope that this "meal" will whet the appetite for further study of the fascinating processes of braiding.

2

THE REGULAR-NESTED CYLINDRICAL BRAIDS

The Standard Herringbone Pineapple Knots belong to the class of the Regular-Nested Cylindrical Braids. It should be noted that, depending on the values of the parameters involved and the coding used, these braids can result in knots having the form of a sphere. An investigation by P. van de Griend into some properties of Spherical Knot Covers has recently been published. (see Ref. [10]).

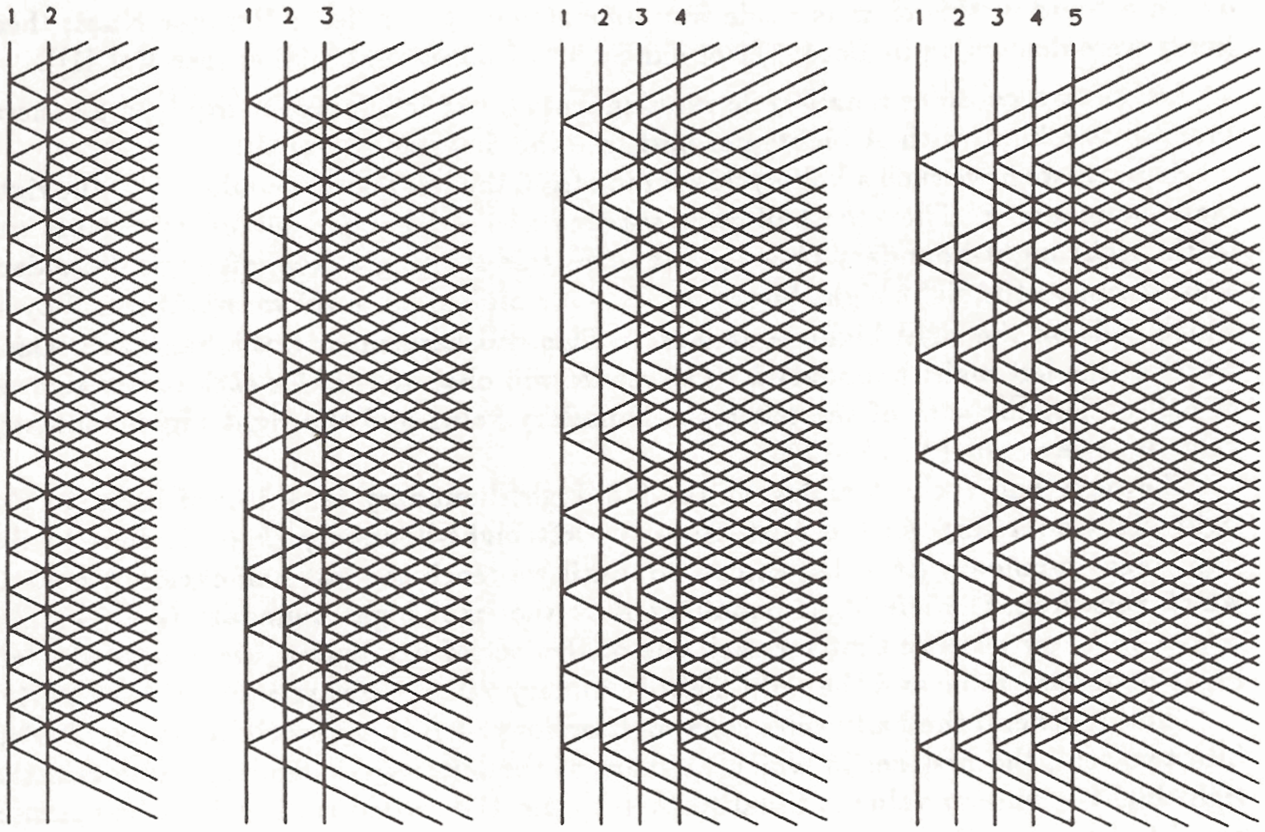
Before we can explain the general construction details of Standard Herringbone Pineapple Knots, we first have to define a Regular-Nested Cylindrical Braid, and discuss three very important parameters that are invariants (constants) for any such braid.

Definition:

> A Regular-Nested Cylindrical Braid is a cylindrical braid in which the circular edges consist of **nests** of bights such that each nest contains the **same number of single bights**, stacked uniformly, as demonstrated by left-edge examples in Fig. 1 and right-edge examples in Fig. 2.
>
> The number of bights per nest will be indicated by A, and the number of nests per circular edge will be indicated by B^*. Hence the total number of bights per circular edge is equal to $A.B^*$, and this will be denoted by B.

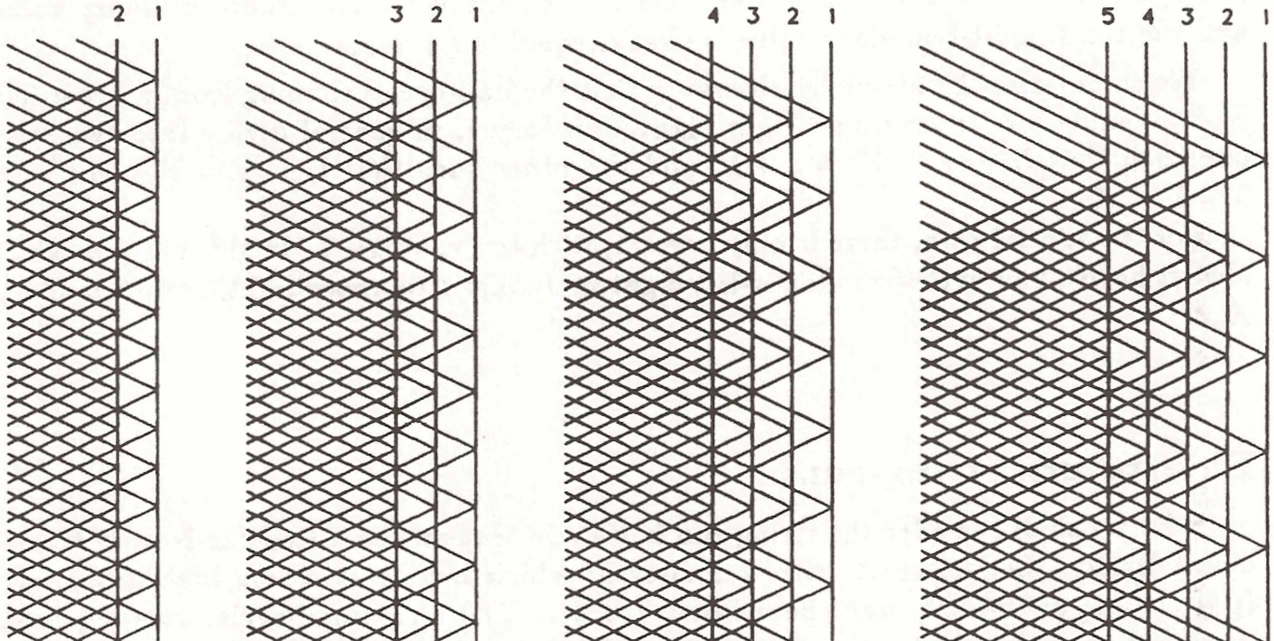
For various values of A, the arrangements of nests are shown, at the left-hand edge in Fig. 1 and at the respective right-hand edge in Fig. 2. (N.B. In these Figures, only portions of the string-run of these Regular-Nested Cylindrical Braids are shown. For examples of fully-drawn Regular-Nested Cylindrical Braids, see Figures 4, 5 and 6.)

Each left- and respective right-hand edge in a complete cylindrical braid of this kind can be seen as a series of A, regular spaced, parallel bight-boundaries, each one having B^* bights. These bight-boundaries may be numbered in sequence from the outside to the inside, as shown in the diagrams.



A = 2 A = 3 A = 4 A = 5

Fig. 1 — Nested Bights at left-edge of Braid.



A = 2 A = 3 A = 4 A = 5

Fig. 2 — Nested Bights at right-edge of Braid.

It should be noted that for $A = 1$ we obtain the class of Regular Cylindrical Braids. When a braid in this class is made from one string, it is called a Regular Knot; these knots were dealt with in Book 1/1 of our Series of books on braiding (see Ref. [1]).

Let us now consider a half-cycle running from lower-left to upper-right, in a general braid of this kind, with A bights per nest (see the diagram in Fig. 3).

Suppose we have such a half-cycle running from the left bight-boundary 1 to the right bight-boundary k. Then the half-cycle starting at the left bight-boundary 2 must run to the right bight-boundary $(k-1)$; the half-cycle starting at the left bight-boundary 3 must run to the right bight-boundary $(k-2)$; and so on, until we reach a half-cycle which ends on the right bight-boundary 1. This will happen with the half-cycle which starts at the left bight-boundary k. The reader will observe that for each one of this set of half-cycles, the sum of the left bight-boundary value and the right bight-boundary value is always equal to $(k+1)$.

The next half-cycle, starting at the left bight-boundary $(k+1)$, will run to the right bight-boundary A; the next, from the left bight-boundary $(k+2)$, runs to the right bight-boundary $(A-1)$, and so on, until we reach the last half-cycle in this set, which runs from the left bight-boundary A to the right bight-boundary $(k+1)$. The reader will now observe that for each one of this set of half-cycles, the sum of the left bight-boundary value and the right bight-boundary value is always equal to $(A+k+1)$.

Thus in general the half-cycles running from lower-left to upper-right can be divided into two sets, the first set in which the sum of the left bight-boundary value and the right bight-boundary value is equal to $(k+1)$ and the second set in which this sum is equal to $(A+k+1)$.

We still have to consider a special case, that in which $k = A$. For this case the second set described above is empty (there are no half-cycles belonging to this set; all half-cycles belong to the first set) and hence the sum of the left bight-boundary value and the right bight-boundary value is always equal to $(A+1)$.

We summarise the above by observing that the half-cycles running from a lower-left bight-boundary l_i to an upper-right bight-boundary r_i in general divide into two sets; one such that $l_i + r_i = Y = k+1$, and the other such that $l_i + r_i = Y + A$, with $2 \leq Y \leq A$.

For the special case, there is only one set, with $l_i + r_i = A+1$. In this case, we shall always have a half-cycle from lower-left bight-boundary 1 to upper-right bight-boundary A.

First-return string-runs

Next we shall consider the string-run half-cycle sequence in a Regular-Nested Cylindrical Braid. Let us start with a half-cycle which begins at a left bight-boundary l_1 (l_1 is one of the left bight-boundaries $1, 2, 3, \ldots, A$). This half-cycle, running from lower-left to upper-right, will end at a right bight-boundary, say r_1 (r_1 is one of the right bight-boundaries $1, 2, 3, \ldots, A$). This half-cycle is followed by a half-cycle (running from lower-right to upper-left) from the right bight-boundary r_1 to the left bight-boundary l_2; next follows a half-cycle running from the left bight-boundary l_2 to the right bight-boundary r_2; then one from r_2 to l_3, followed by one from l_3 to r_3, and so on.

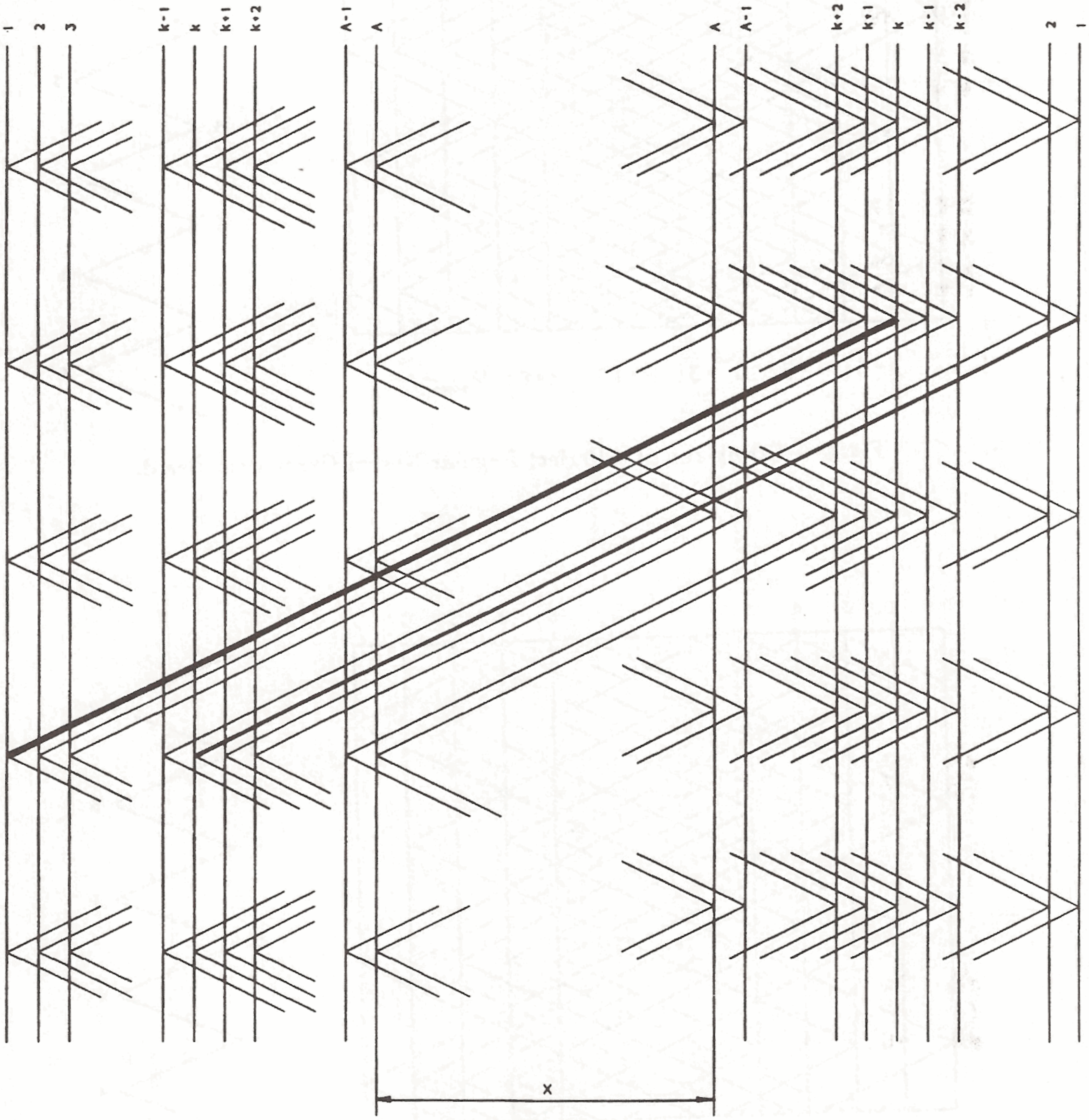


Fig. 3 — Half-cycles between left Bight-boundaries and right Bight-boundaries.

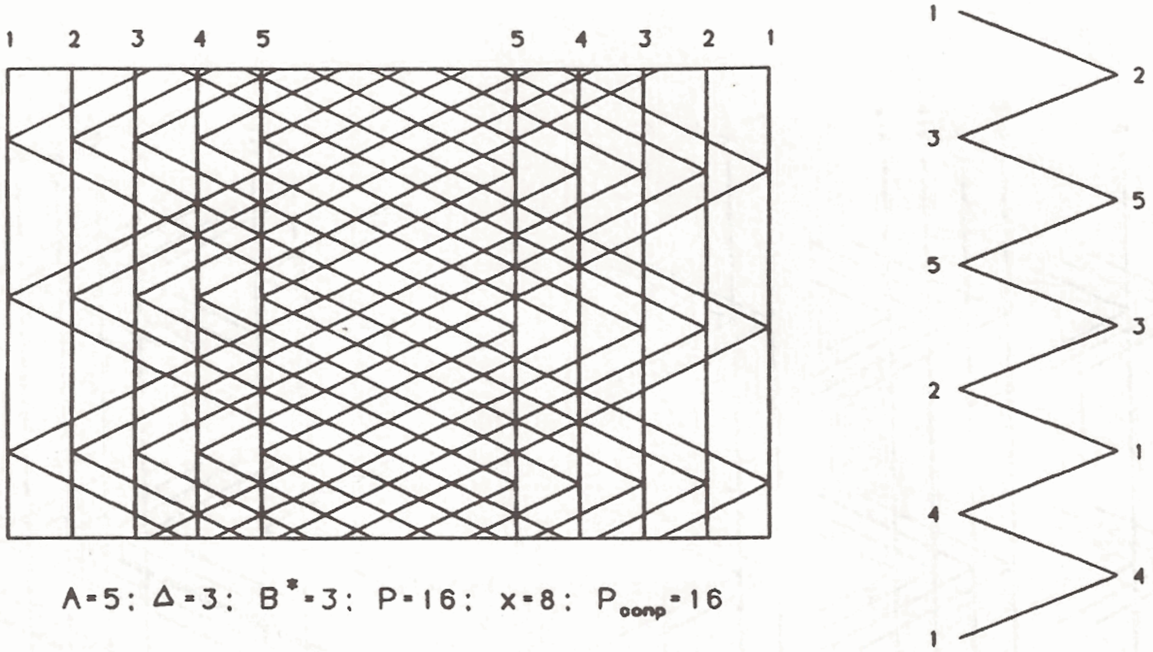


Fig. 4 — String-run of a Perfect Regular-Nested Cylindrical Braid.

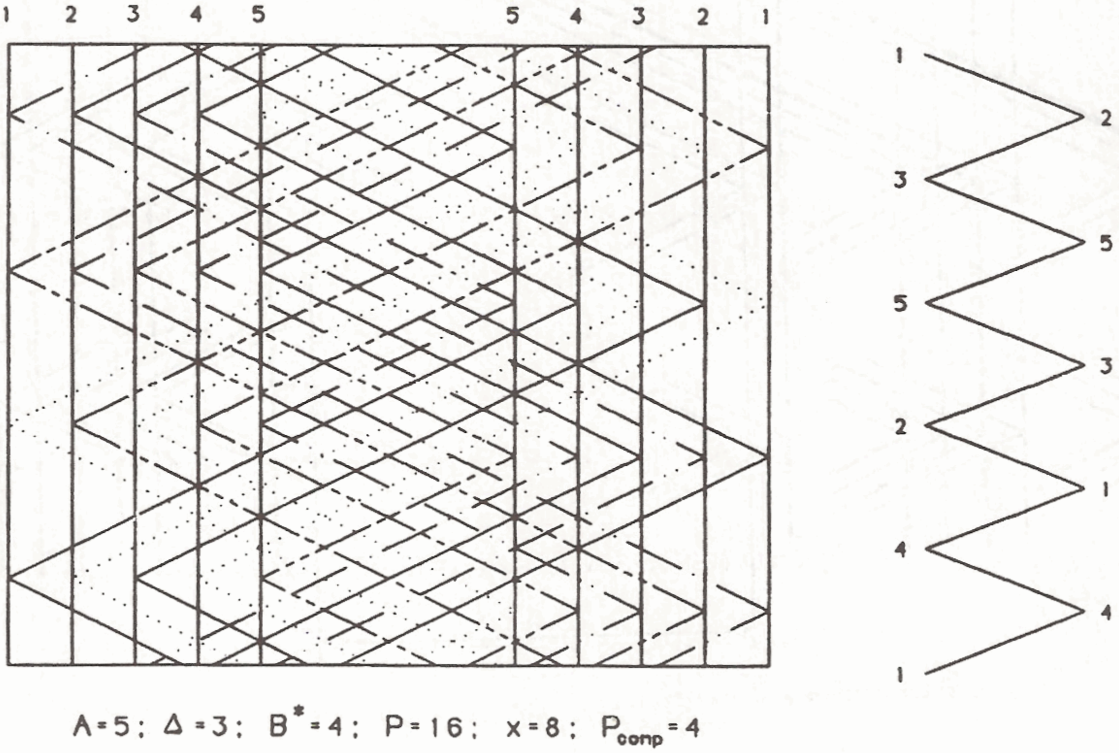


Fig. 5 — String-run of a Semi-Perfect Regular-Nested Cylindrical Braid.

Eventually we return, with a right to left half-cycle, to our initial left bight-boundary l_1. This does not mean, of course, that we necessarily return to our starting point, since we have B^* bights on each bight-boundary (see the examples of Figs. 4 and 5).

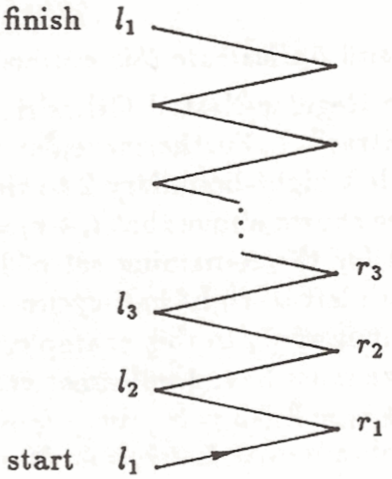
A partial string-run, as described here, we shall call a 'first-return string-run', which we define as follows:

Definition:

> A **first-return string-run** is a series of consecutive cycles which begins on a given **left** bight-boundary (k say) and ends when the **left** bight-boundary k is first reached again.

The **total** string-run for the braid is now seen to be a sequence of first-return string-runs. This sequence in some cases forms a single-string braid; whereas in others, an interwoven braid of several strings results.

The above defined first-return string-run can be shown graphically as follows:

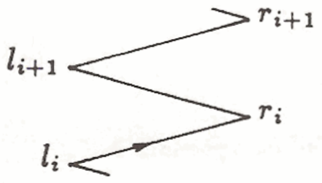


We have now obtained a sequence of left bight-boundaries $l_1, l_2, l_3, \ldots, l_1$ and a sequence of right bight-boundaries $r_1, r_2, r_3, \ldots$ (The reader should trace a sequence of half-cycles in the single-string knot of Fig. 4, in order to learn how the l_i and r_i occur in relation to one another on the two edges containing the bight-boundaries.)

In these bight-boundary sequences a parameter called Δ (i.e. upper-case Delta) plays an important role. This parameter is an invariant (has a constant value) for a given Regular-Nested Cylindrical Braid.

With the aid of the parameter Δ we can express the value of l_2 in terms of that of l_1, the value of l_3 in that of l_2, etc.; the value of r_2 in that of r_1, the value of r_3 in that of r_2, etc.

To show how this is effected, let us first take any three consecutive half-cycles; these we can represent graphically as follows:



wherein i is chosen from $\{1, 2, 3, \ldots, (A-1)\}$

Then[†]:

$$l_{i+1} = |l_i + \Delta|_A \qquad \text{and} \qquad r_{i+1} = |r_i - \Delta|_A$$

The value of Δ, for any given knot, is one of the values $0, 1, 2, \ldots, (A-2), (A-1)$; moreover, this Δ-value is the same in each cycle.

Thus in the general first-return string-run we obtain:

$$\begin{array}{ll}
l_2 = |l_1 + \Delta|_A & r_2 = |r_1 - \Delta|_A \\
l_3 = |l_2 + \Delta|_A & r_3 = |r_2 - \Delta|_A \\
l_4 = |l_3 + \Delta|_A & r_4 = |r_3 - \Delta|_A \\
\vdots & \vdots \\
l_{i+1} = |l_i + \Delta|_A & r_{i+1} = |r_i - \Delta|_A \\
\vdots & \vdots
\end{array}$$

These calculations cease when l_1 is first reached again.

Example 1: *Figs. 4 and 5 illustrate this example.*

Suppose we consider the Regular-Nested Cylindrical Braid with $A = 5$ and $\Delta = 3$ (we choose these values arbitrarily). Furthermore, let the first-return string-run have a left-to-right half-cycle from left bight-boundary 2 to right bight-boundary 3 (again, the choice is arbitrary). We have shown above that $l_i + r_i = k + 1$ for one set of left-to-right half-cycles and $(A + k + 1)$ for the remaining set of left-to-right half-cycles, or if the special case $k = A$ applies, all left-to-right half-cycles are such that $l_i + r_i = A + 1$.

This last special case cannot apply in this example, since $l_i + r_i = 2 + 3 = 5$ whereas $A + 1 = 5 + 1 = 6$. Hence we must have the former case.

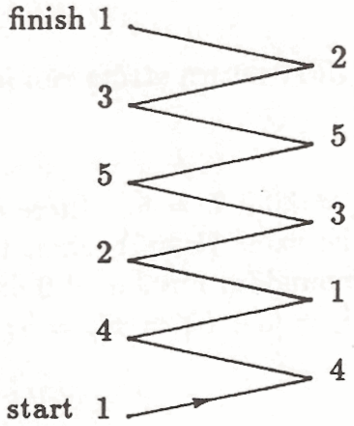
Thus for this braid $l_i + r_i = 2 + 3 = 5$ or $l_i + r_i = A + 2 + 3 = 5 + 2 + 3 = 10$.

For $l_1 = 1$ we therefore obtain $r_1 = 5 - 1 = 4$. Now we can readily write down the l_i and r_i sequences as follows:

$$\begin{array}{ll}
l_1 = 1 & r_1 = 4 \\
l_2 = |1 + \Delta|_5 = |1 + 3|_5 = 4 & r_2 = |4 - \Delta|_5 = |4 - 3|_5 = |1|_5 = 1 \\
l_3 = |4 + \Delta|_5 = |4 + 3|_5 = |7|_5 = 2 & r_3 = |1 - \Delta|_5 = |1 - 3|_5 = |-2|_5 = 3 \\
l_4 = |2 + \Delta|_5 = |2 + 3|_5 = |5|_5 = 5 & r_4 = |3 - \Delta|_5 = |3 - 3|_5 = |0|_5 = 5 \\
l_5 = |5 + \Delta|_5 = |5 + 3|_5 = |8|_5 = 3 & r_5 = |5 - \Delta|_5 = |5 - 3|_5 = |2|_5 = 2 \\
l_6 = |3 + \Delta|_5 = |3 + 3|_5 = |6|_5 = 1 = l_1 &
\end{array}$$

Graphically we obtain the following first-return string-run:

[†] *Here we use the modulus notation, which is defined and exemplified in Appendix 3.*



Example 2: *Fig. 6 illustrates this example.*

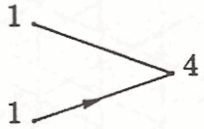
This time we consider the Regular-Nested Cylindrical Braid with $A = 5$ and $\Delta = 0$. Furthermore, let the first-return string-run again have a left-to-right half-cycle from left bight-boundary 2 to right bight-boundary 3.

As for Example 1, we obtain $l_i + r_i = 2+3 = 5$ or $l_i + r_i = A+2+3 = 5+2+3 = 10$.

If we start at $l_1 = 1$ then we obtain for r_1 the value $5 - 1 = 4$.

Since $\Delta = 0$ we then obtain $l_2 = |1+0|_5 = |1|_5 = 1$; hence we arrive back at the initial left bight-boundary 1.

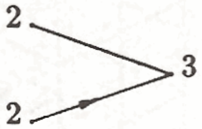
Thus the initial first-return string-run is



For $l_1 = 2$ we obtain $r_1 = 5 - 2 = 3$.

Since $\Delta = 0$ we obtain $l_2 = |2+0|_5 = |2|_5 = 2$; hence we arrive back at the initial left bight-boundary 2.

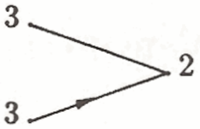
Thus the 2^{nd} first-return string-run is



For $l_1 = 3$ we obtain $r_1 = 5 - 3 = 2$.

Since $\Delta = 0$ we obtain $l_2 = |3+0|_5 = |3|_5 = 3$; hence we arrive back at the initial left bight-boundary 3.

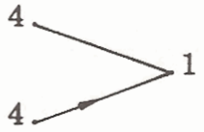
Thus the 3^{rd} first-return string-run is



For $l_1 = 4$ we obtain $r_1 = 5 - 4 = 1$.

Since $\Delta = 0$ we obtain $l_2 = |4+0|_5 = |4|_5 = 4$; hence we arrive back at the initial left bight-boundary 4.

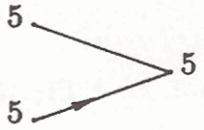
Thus the 4th first-return string-run is



For $l_1 = 5$ we obtain $r_1 = 10 - 5 = 5$ (here we have to use the value $l_i + r_i = A + k + 1 = 10$ instead of the value $(k + 1)$, since the value $k + 1 = 5$ would result in $r_1 = 5 - 5 = 0$ and a bight-boundary numbered 0 does not exist).

Since $\Delta = 0$, we obtain $l_2 = |5 + 0|_5 = |5|_5 = 5$; hence we arrive back at the initial left bight-boundary 5.

Thus the 5th first-return string-run is



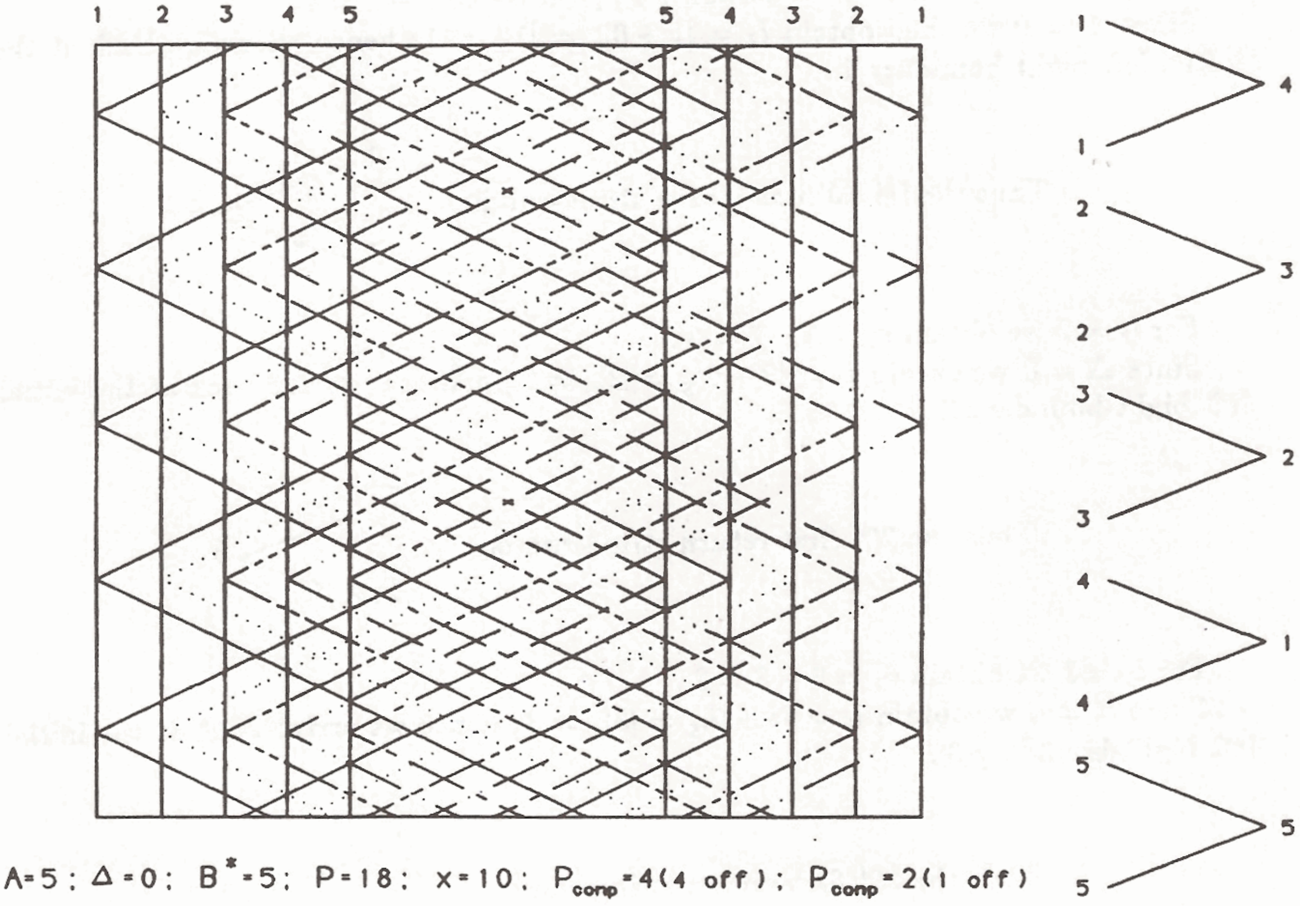

1 2 3 4 5 5 4 3 2 1
1
4
1
2
3
2
3
2
3
4
1
4
5
5
5
A=5 : Δ=0: B*=5: P=18: X=10: P_comp=4(4 off): P_comp=2(1 off)

Fig. 6 — String-run of a Standard Regular-Nested Cylindrical Braid.

In a Regular-Nested Cylindrical Braid the innermost left and right bight-boundaries are a distance of x columns or x parts apart (see Fig. 3). The value of x has to be specified by the braider, before the grid-diagram of the braid can be drawn.

Once the value of x has been specified, all the A bight-boundaries on each of the left and right edges respectively are determined, since two adjacent bight-boundaries (belonging to the same edge) are always two columns apart.

Note: the concept of columns on grid-diagrams is dealt with in Book 1/1 (see Ref. [1]). Columns are vertical lines through crossing points.

We can show that x can always be expressed as $x = cA + |2(l_i + r_i) + \Delta|_A$, where c is some integer. We shall find it convenient to label the quantity $|2(l_i + r_i) + \Delta|_A$ by the symbol δ (delta), where δ is one of the integers $1, 2, 3, \ldots, (A-1), A$.

It can be shown that for a given Regular-Nested Cylindrical Braid, both c and δ are invariants (i.e. they do not vary within a given knot).

The total number of parts in a Regular-Nested Cylindrical Braid is given by P, where $P = x + 2(A-1) = x + 2A - 2$ as can be quickly determined from Fig. 3.

Sub-classes of the Regular-Nested Cylindrical Braids

The class of the Regular-Nested Cylindrical Braids can be divided into six sub-classes. In general we have in each of these sub-classes mirror-image pairs indicated by Type 1 and Type 2. These are defined within a sub-class as follows:

$$\begin{aligned} &\textit{Type}\,1: \quad \Delta = \Delta_1\,, \qquad \text{where} \quad 1 \le \Delta_1 < A/2 \\ &\textit{Type}\,2: \quad \Delta = \Delta_2 = A - \Delta_1 \end{aligned}$$

Type 1 and Type 2 braids have the same Δ value when $\Delta = A/2$ (A has to be even for these braids).

Suppose that Type 1 has a half-cycle from lower-left to upper-right, running from the left bight-boundary l_{i_1} to the right bight-boundary r_{i_1}; and that Type 2 has a half-cycle from lower-left to upper-right, running from the left bight-boundary l_{i_2} to the right bight-boundary r_{i_2}. When $l_{i_1} = l_{i_2}$ the following relationships hold:

$$\begin{aligned} r_{i_2} &= |r_{i_1} + \Delta_1|_A & \qquad r_{i_1} &= |r_{i_2} + \Delta_2|_A \\ r_{i_2} &= |r_{i_1} + A - \Delta_2|_A & r_{i_1} &= |r_{i_2} - \Delta_1 + A|_A \\ &= |r_{i_1} - \Delta_2|_A & &= |r_{i_2} - \Delta_1|_A \end{aligned}$$

In the case $\Delta = 0$, then Type 1 is identical to Type 2.
In the case $\Delta_1 = \Delta_2 = A/2$, then:

$$\begin{aligned} r_{i_2} &= |r_{i_1} - A/2|_A = |r_{i_1} + A/2|_A \\ r_{i_1} &= |r_{i_2} - A/2|_A = |r_{i_2} + A/2|_A \end{aligned}$$

A brief summary of the classification of the Regular-Nested Cylindrical Braids is given in Appendix 1.

3

THE HERRINGBONE PINEAPPLE KNOTS

We now present a definition of **Pineapple Knots**. The reader should consult Figures 7 and 8, to help in understanding the details involved.

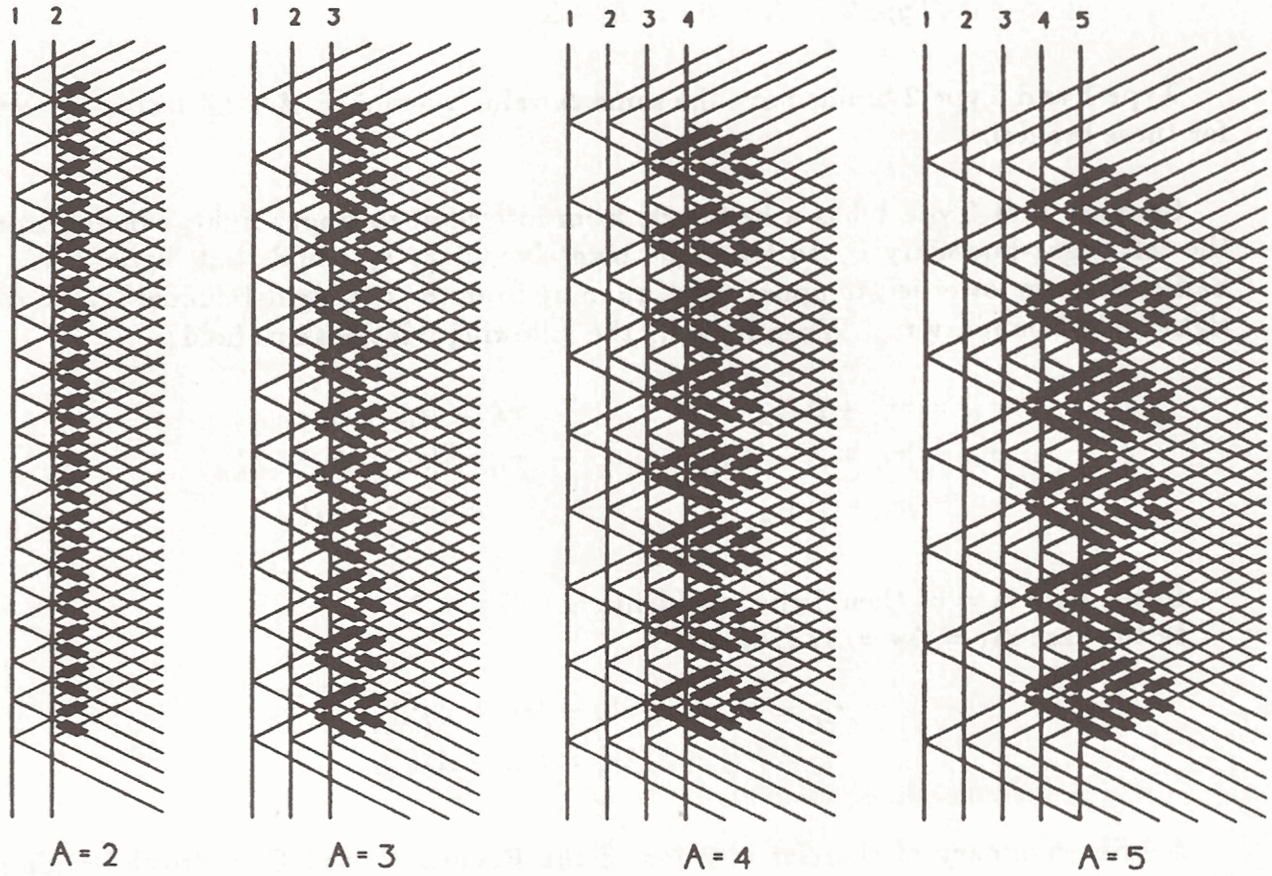


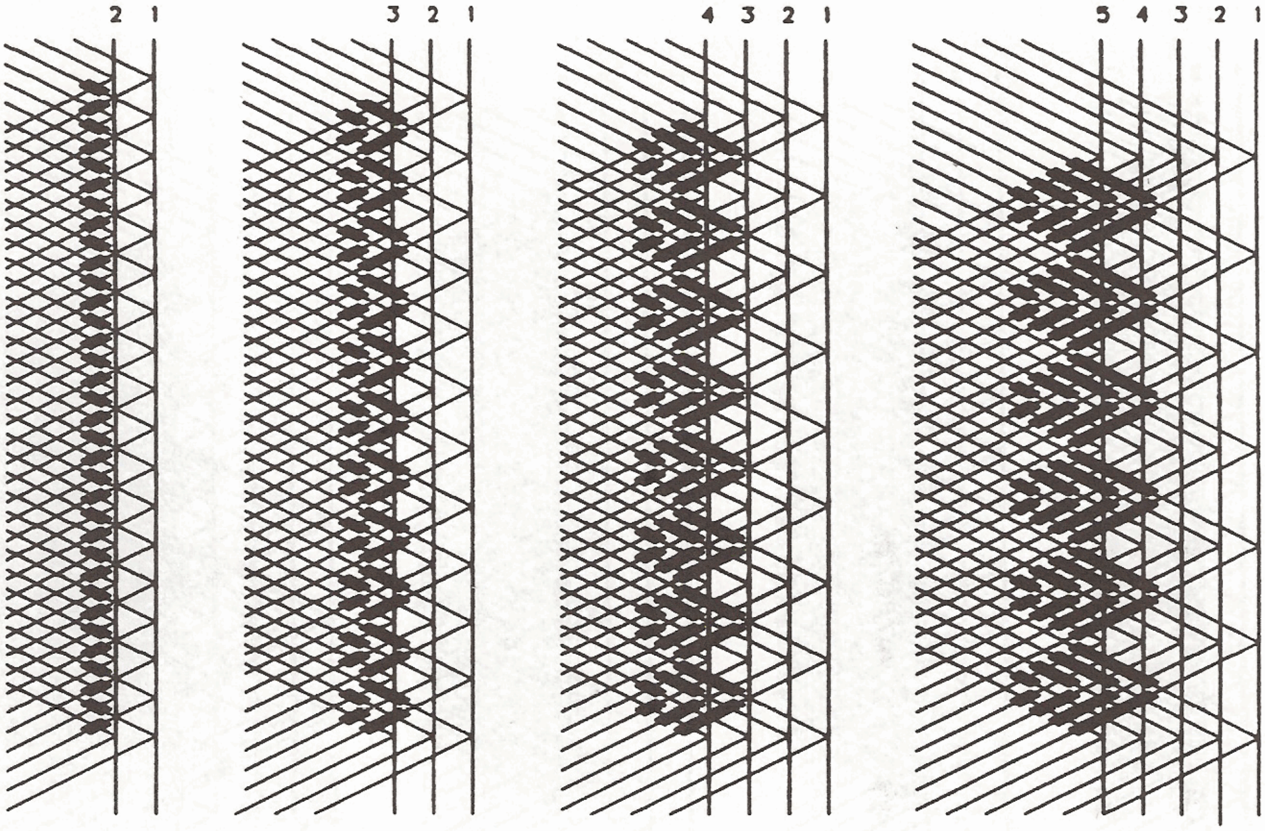
Fig. 7 — The essential coding at the left-edge of Pineapple Knots.

Consider the two half-cycles (one slanting upwards and the other downwards) which issue from a bight-point on bight-boundary k. On these half-cycles, the first $(k-1)$ intersections they each make (with half-cycles issuing from bight-boundaries 1 to $(k-1)$ inclusive) are all **under-crossings**.

This **essential coding** is the crux of the definition for knots of the Pineapple Knot class. In practice, this weave is one which creates cylindrical braids with hemi-spherical ends.

The reader should note that, so far, only the coding (the 'weave pattern') at the edges (i.e. with respect to the nested bights) has been defined. Moreover, even within this region, there are still some undefined crossings. These are the first crossings between the half-cycles from adjacent bight-points on the same bight-boundary. In other words, they are the first 'valley-points' formed by half-cycles from adjacent bight-points on the same bight-boundaries.

Thus, the Pineapple Knots form a knot-class of the class of Regular-Nested Cylindrical Braids, in which the nested bights have coding arrangements as outlined above and which are demonstrated for $A = 2$ to 5 in the diagrams of Figs. 7 and 8.



A = 2 A = 3 A = 4 A = 5

Fig. 8 — The essential coding at the right-edge of Pineapple Knots.

The knot-class of Pineapple Knots can be divided into many sub-classes, one of which is the Herringbone Pineapple Knot sub-class.

The general definition of **herringbone coding** is as follows:

Herringbone coding is a row-coding (see [1], p.45); it consists of sets, each of n adjacent rows, every row within a set having the same coding; the codings in adjacent sets alternate throughout.

The overall effect in a Herringbone coded knot is of a succession of stacked 'V's, arranged with their apices pointing towards the left and right bight-edges.

The Herringbone Pineapple Knot sub-class in turn can be divided into two major families, one which has a herringbone coding, and the other which has a broken-herringbone coding.

The members of the family which have a herringbone coding can be divided into four sub-families; whereas those in the family with broken-herringbone coding can be divided into six sub-families. A short summary of the essential properties of all these sub-families is given in Appendix 2.

The **essential** coding arrangement of the four sub-families which all have a herringbone coding is shown for $A = 2$ to 5 in Figs. 9 and 10. The reader will observe that in every nest there are two rows of crossings which can have a coding as given in Figs. 11 or 12 and Figs. 13 or 14. The resulting herringbone codings near the bight-edges are therefore as given in Figs. 15 or 16 and Figs. 17 or 18. No others are possible, due to the row-coding within sets, as defined above for herringbone coding.

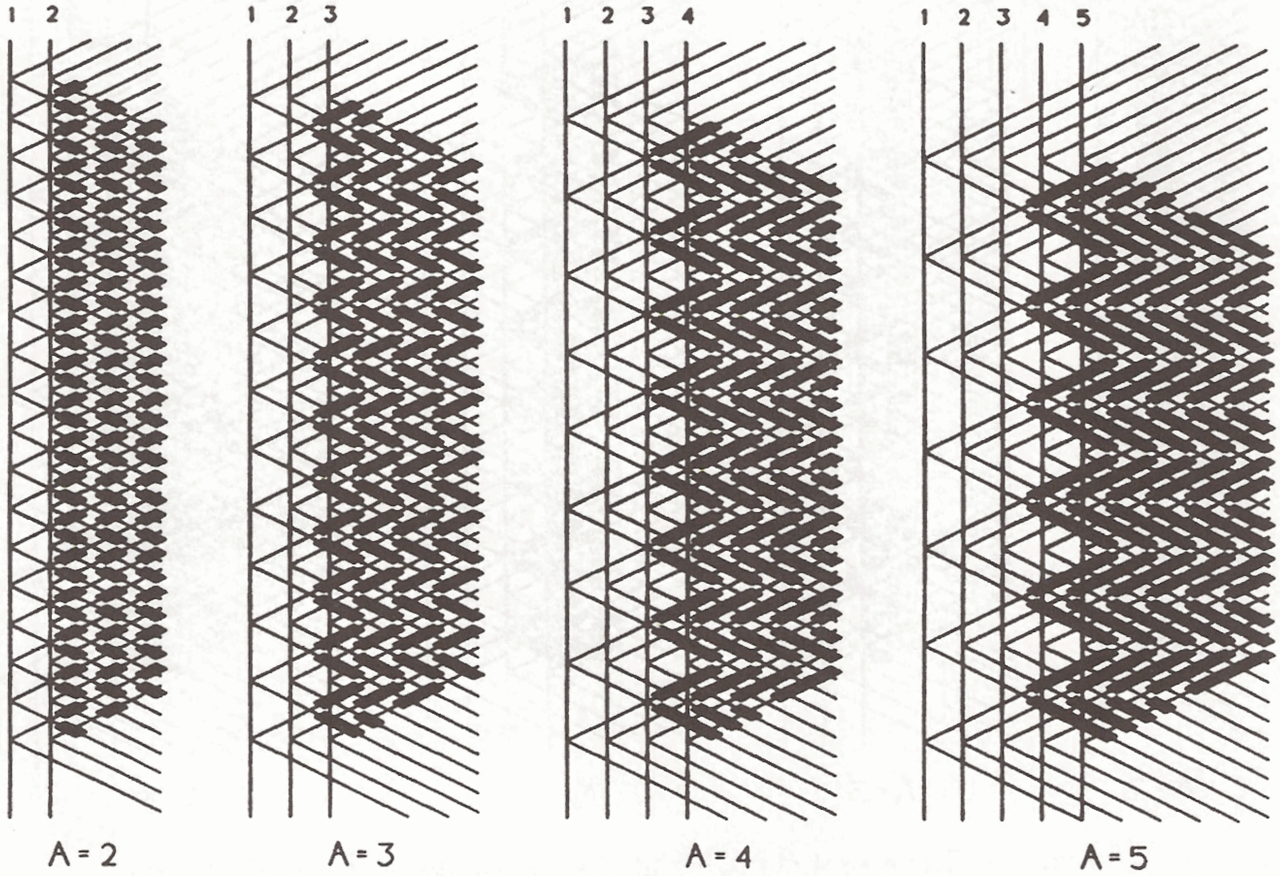


Fig. 9 — The essential coding at the left-edge of herringbone coded Pineapple Knots.

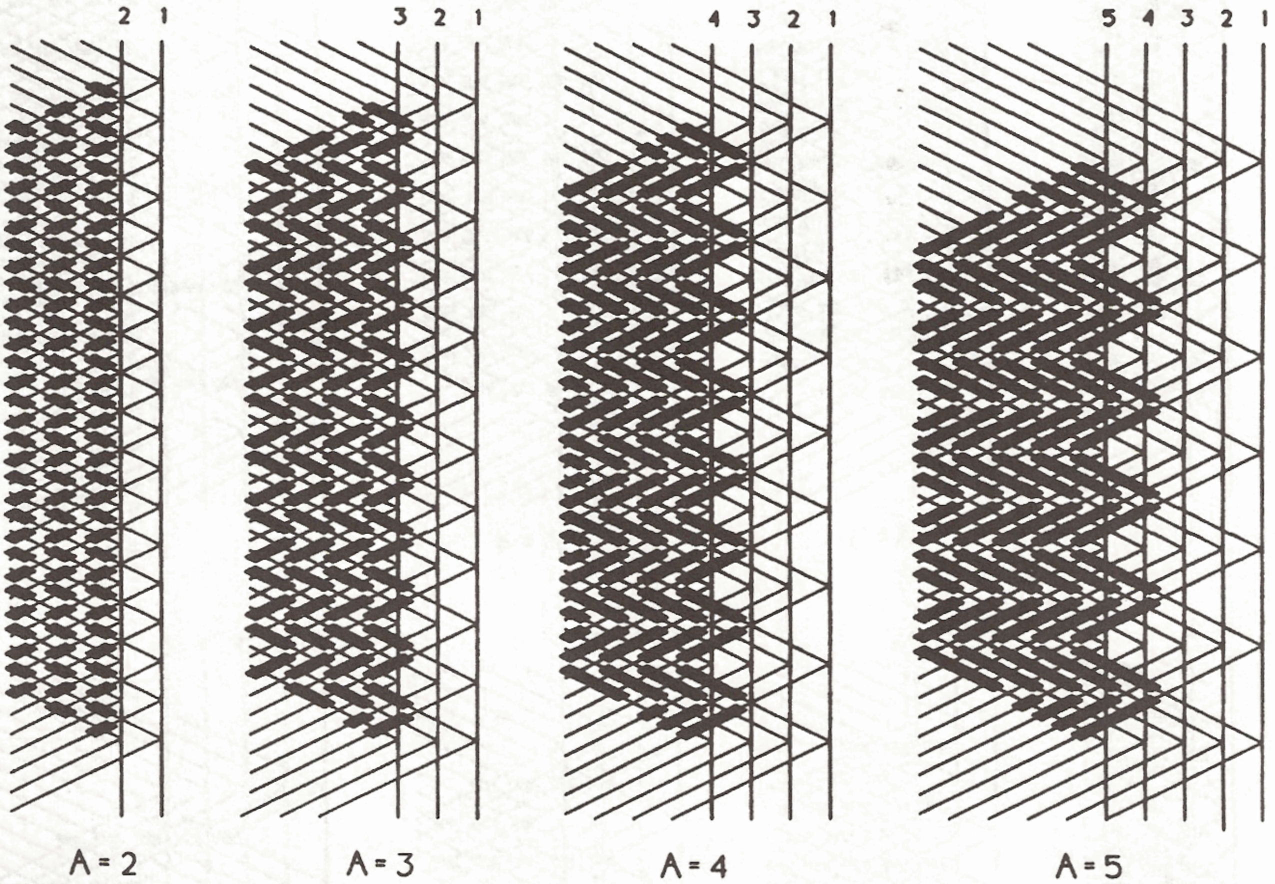
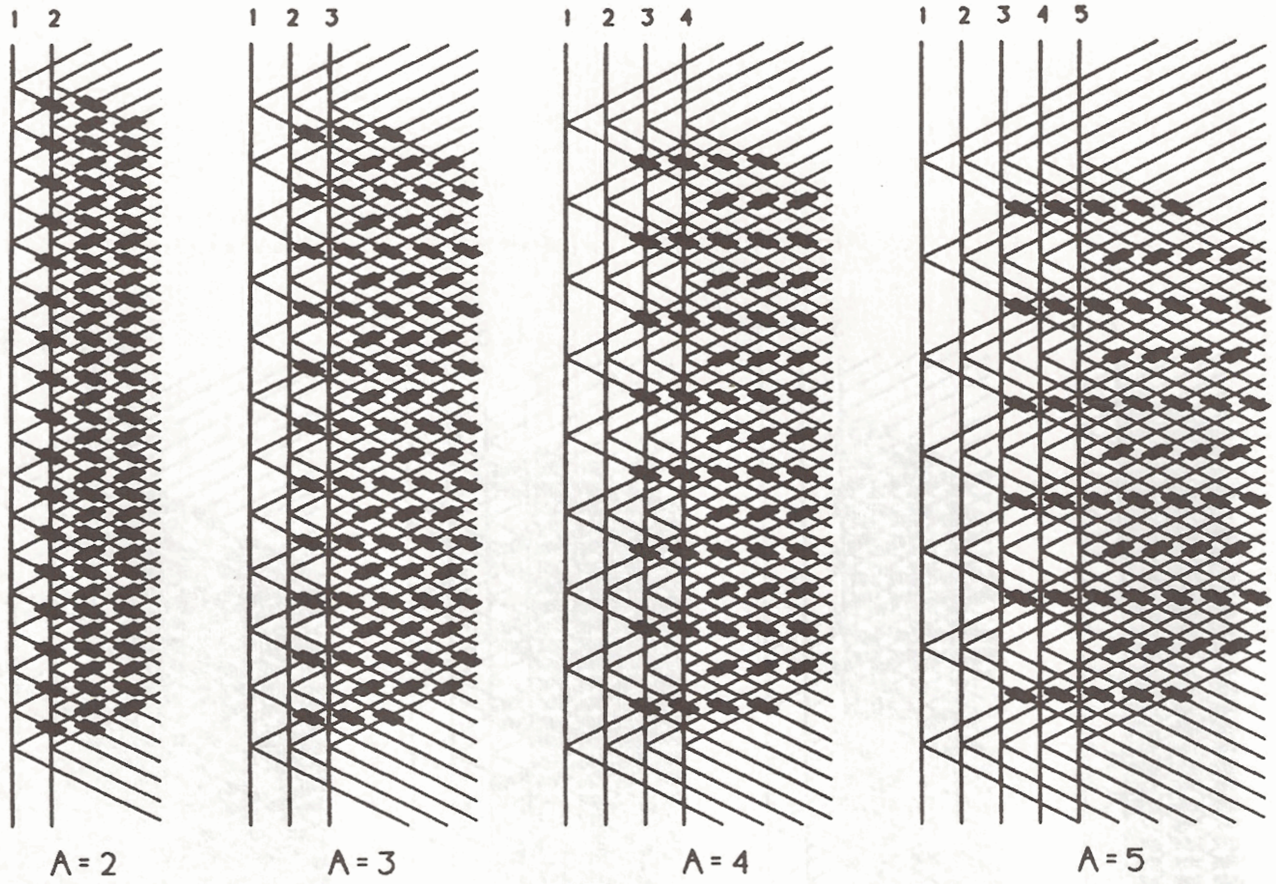


Fig. 10 — The essential coding at the right-edge of herringbone coded Pineapple Knots.



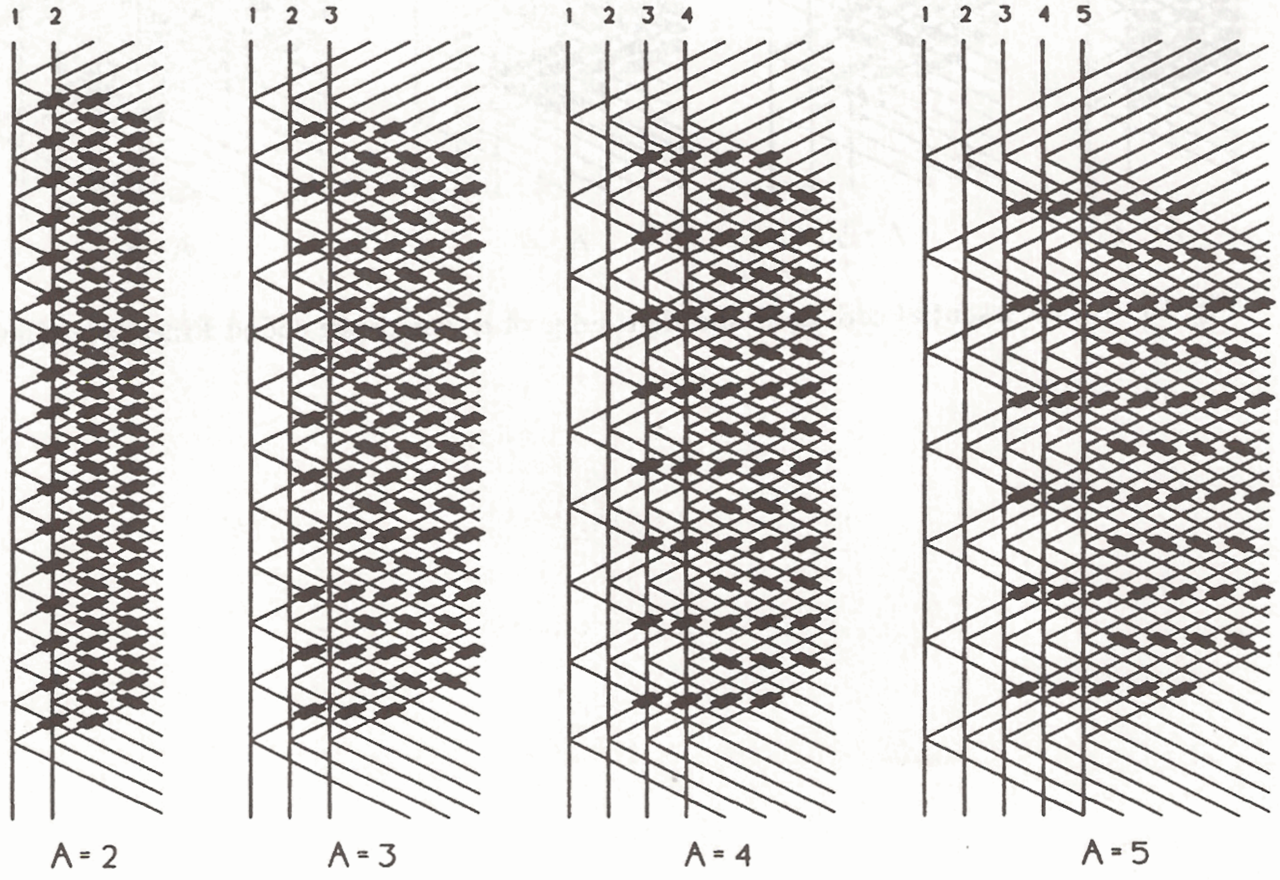
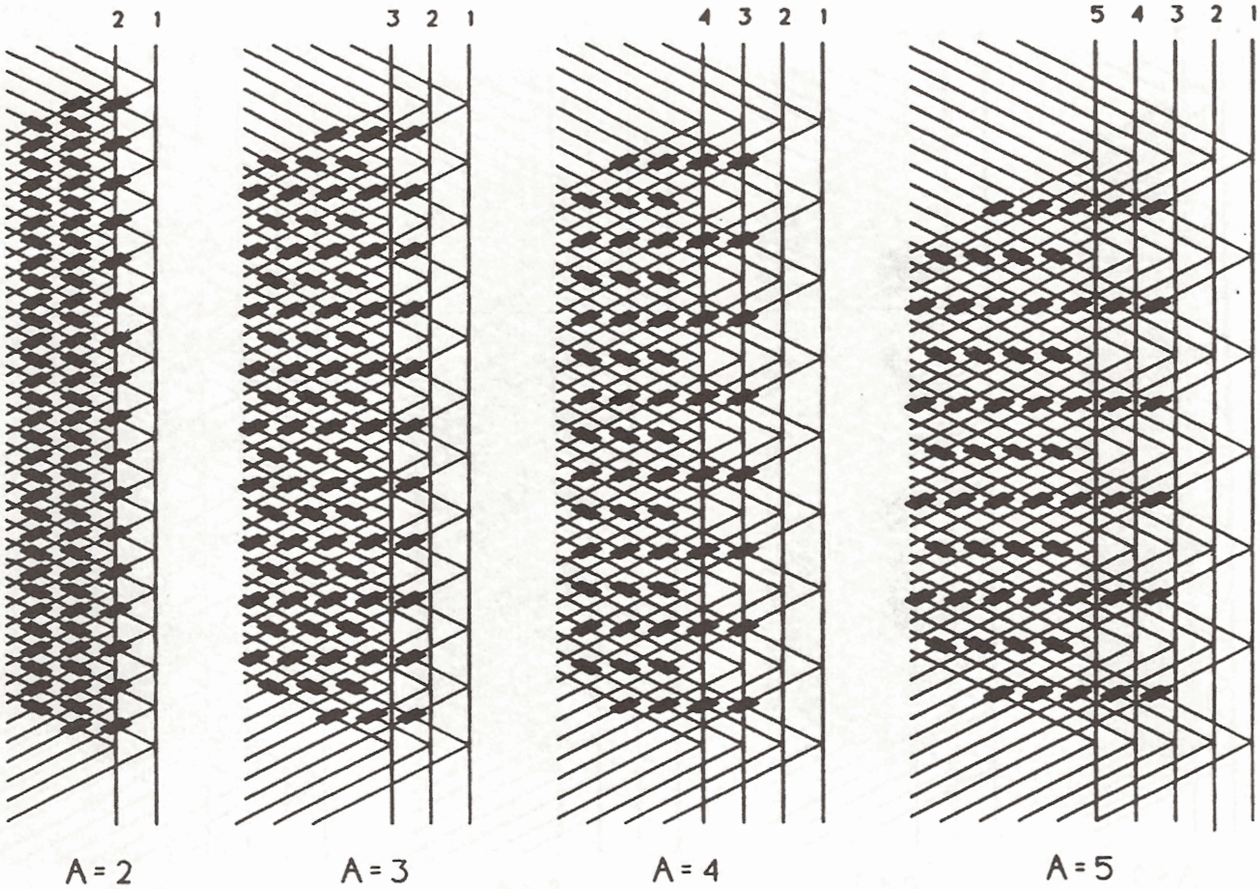


Fig. 11 (top) — The two coding possibilities for the remaining uncoded rows
Fig. 12 (bottom) — at the left-edge of a herringbone coded Pineapple Knot.



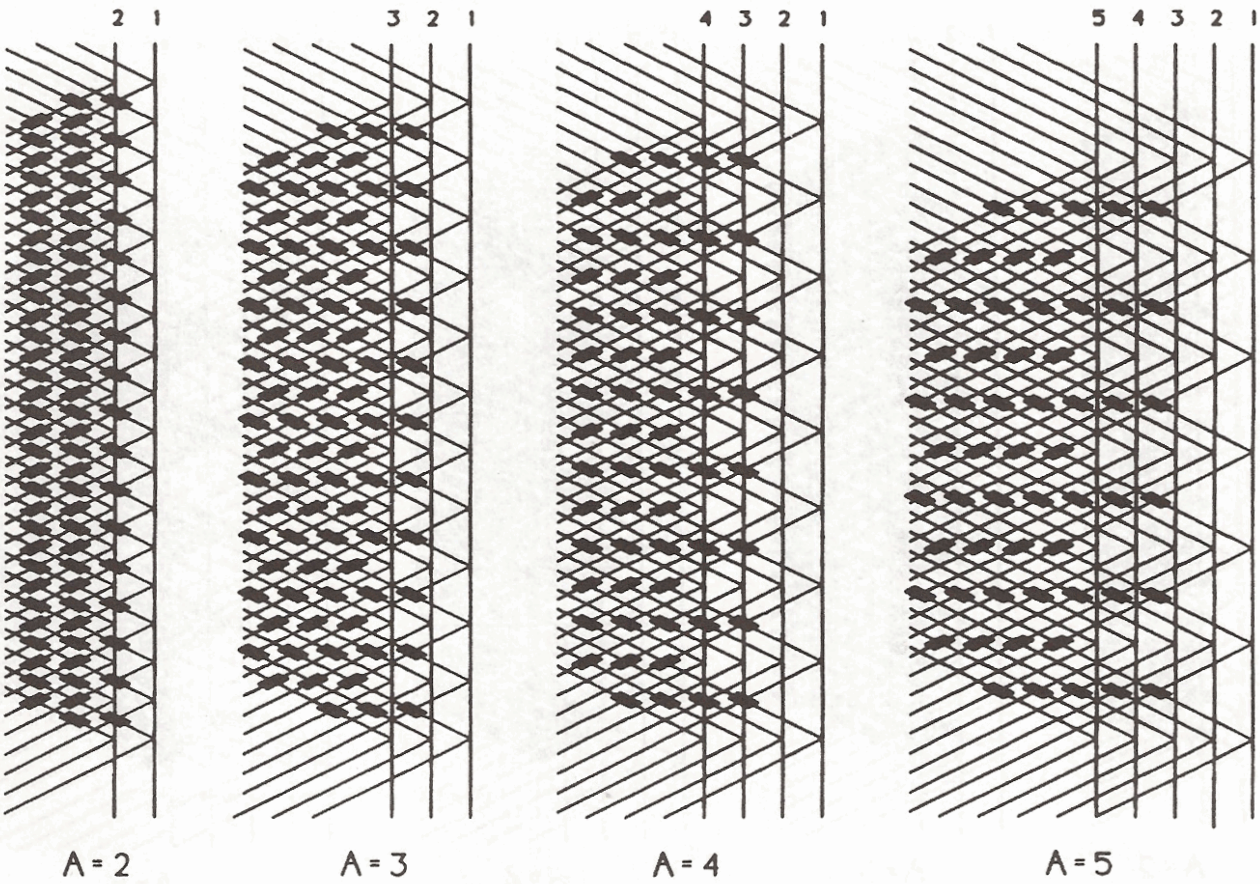
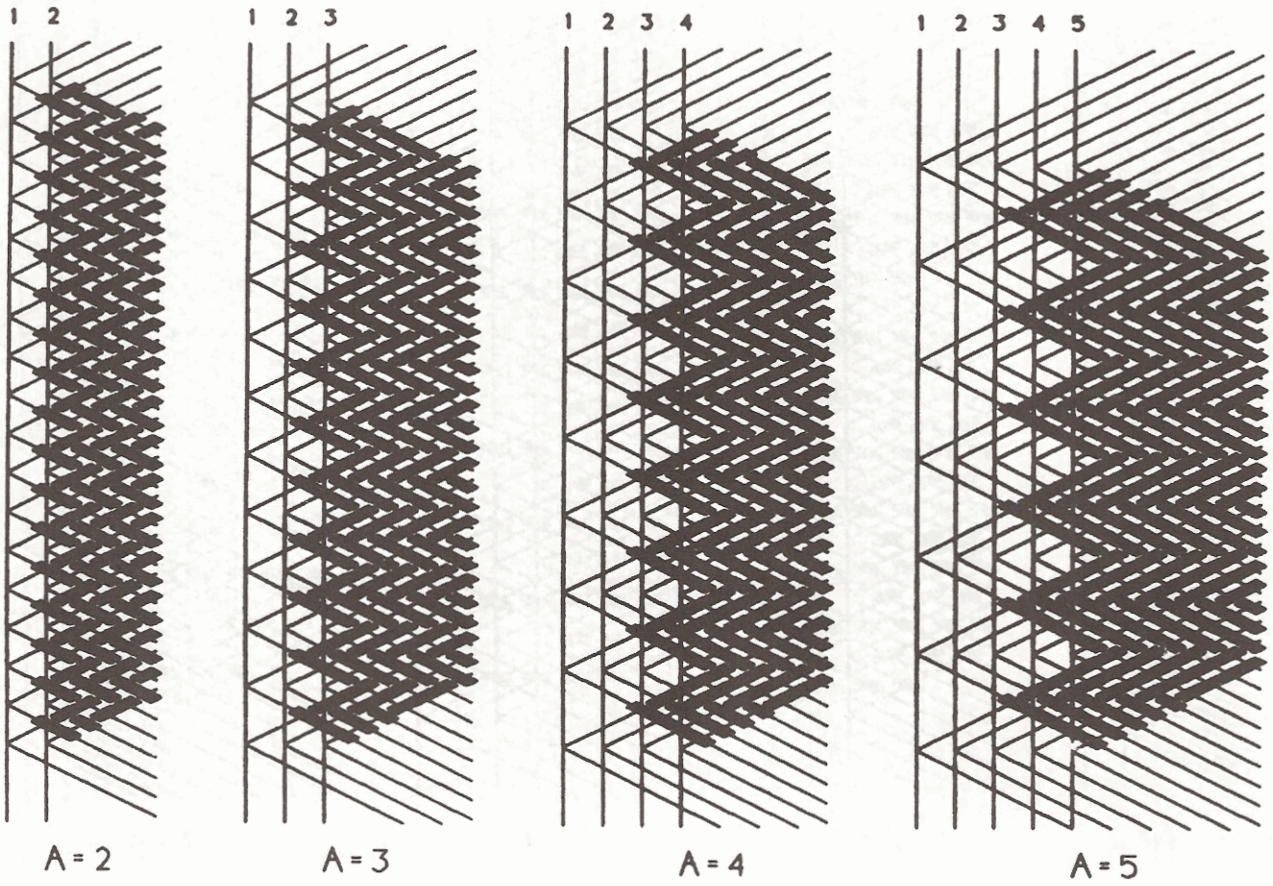


Fig. 13 (top) — The two coding possibilities for the remaining uncoded rows
Fig. 14 (bottom) at the right-edge of a herringbone coded Pineapple Knot.



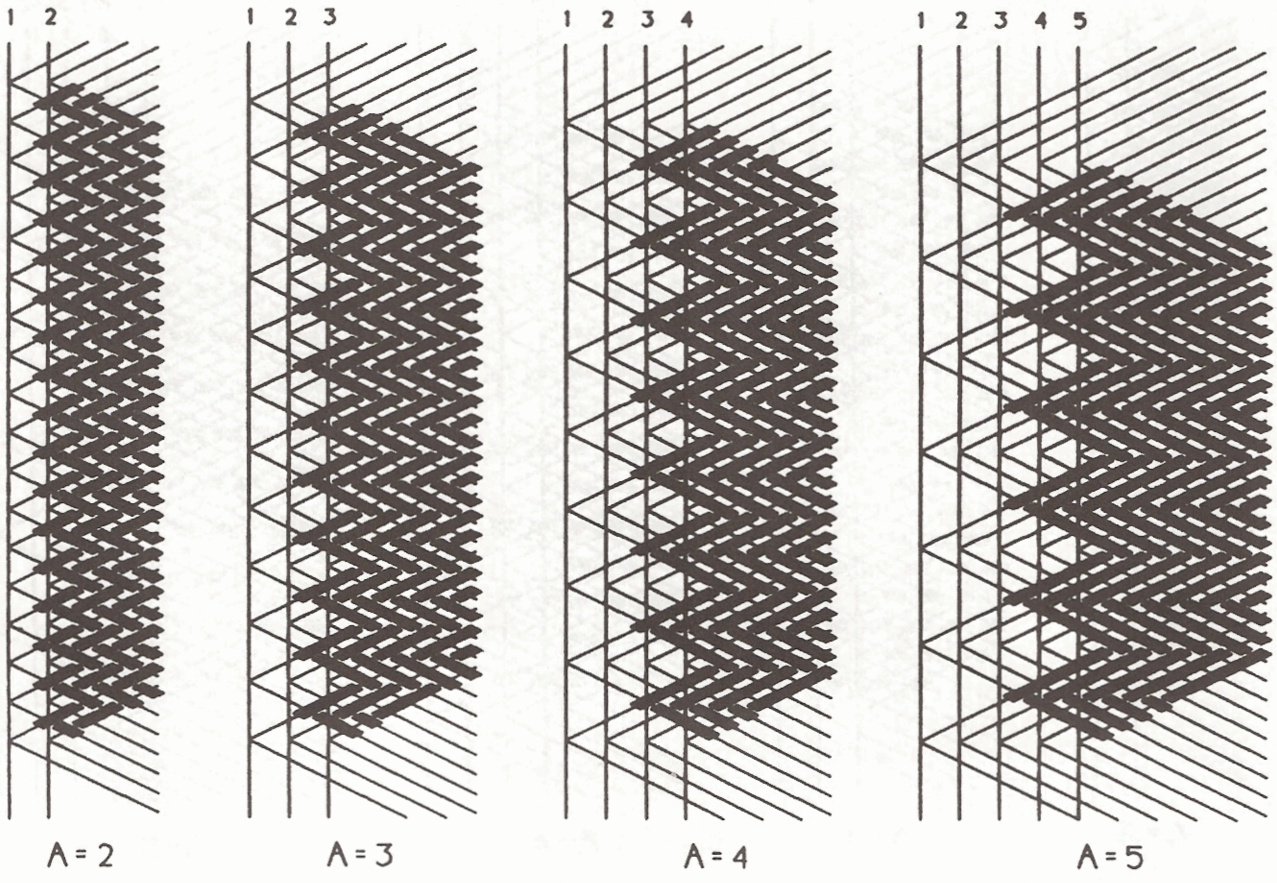
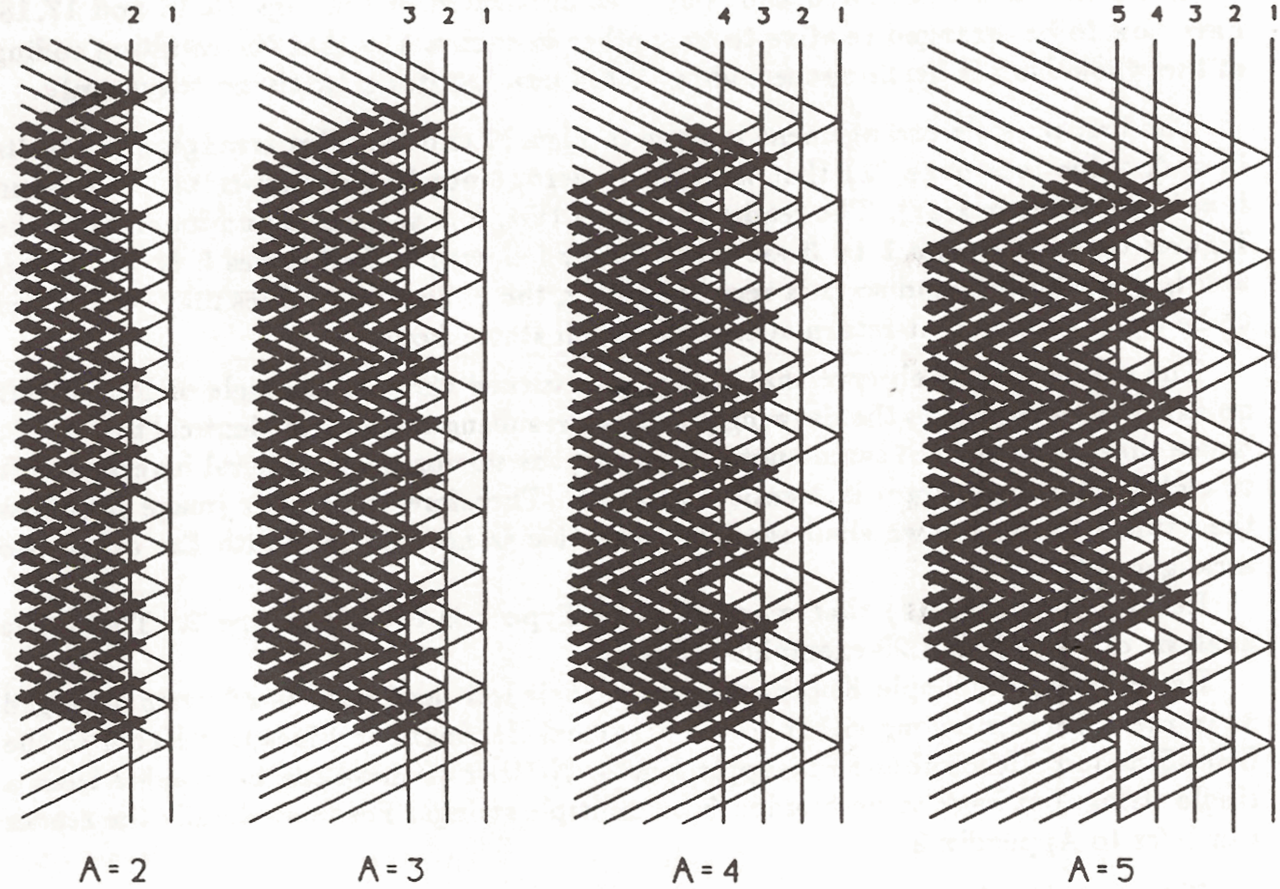


Fig. 15 (top) — The two possible herringbone codings
Fig. 16 (bottom) at the left-edge of a herringbone coded Pineapple Knot.



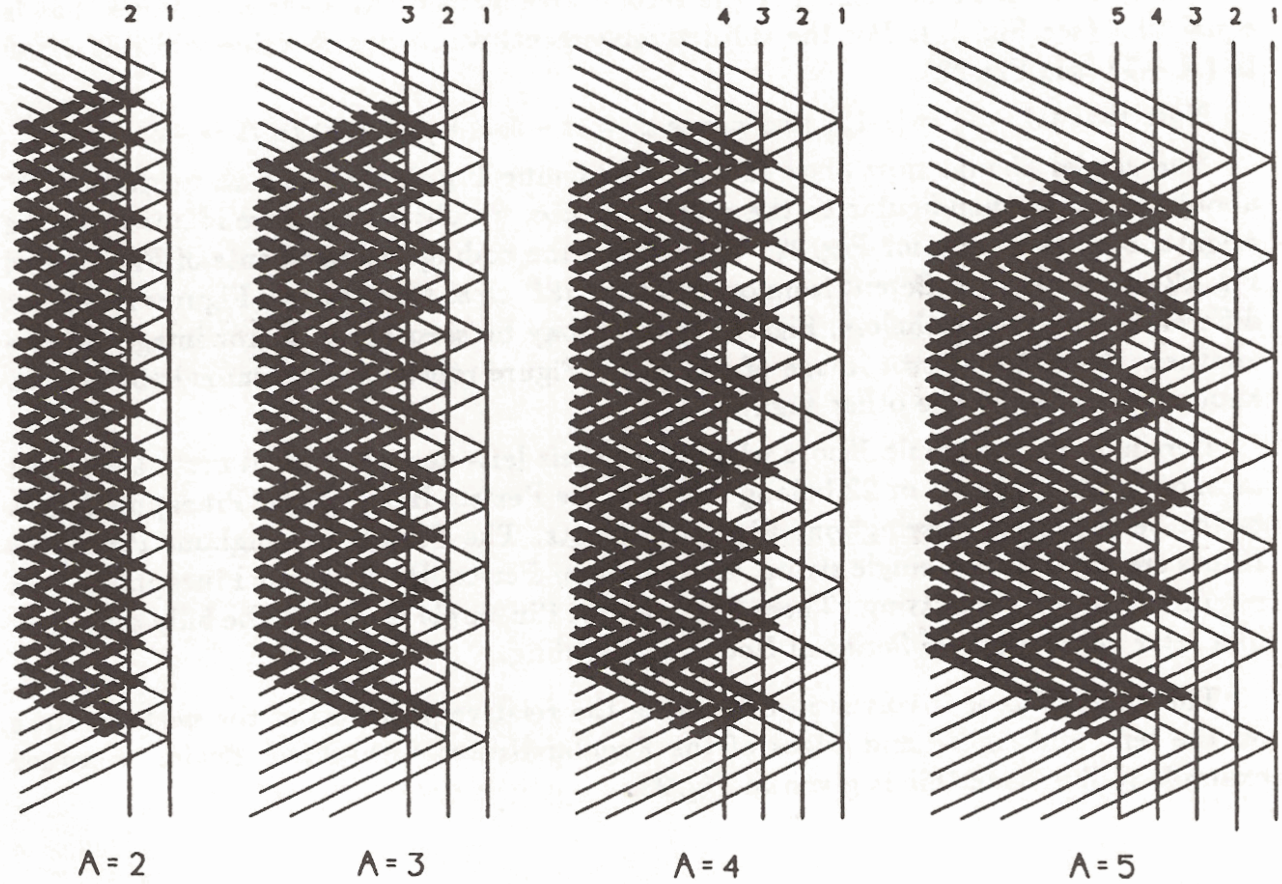


Fig. 17 (top) — The two possible herringbone codings
Fig. 18 (bottom) at the right-edge of a herringbone coded Pineapple Knot.

These left- and right-hand knot parts, as illustrated in the Figs. 15, 16 and 17, 18, have now to be arranged relative to each other in such a way that the resulting coding of the whole knot is herringbone coding. **This can be done in three ways only.**

The first possible arrangement is given in Figs. 19 and 20. This arrangement results in a Δ value (see page 11) that is equal to zero, since in any first-return string-run $l_i = l_{i+1}$ (also $r_i = r_{i+1}$). The reader can check this, for example, by noting in both the Figures that half-cycles **1 to 3** are immediately followed by half-cycles **3 to 1**; here, l_i and l_{i+1} are both 1, and so Δ is zero. Similarly, the r_i and r_{i+1} values may be checked to be equal for any first-return string-run which starts from r_i.

The reader should observe that if Fig. 19 is turned through an angle of 180° about an axis perpendicular to the drawing plane, the resulting diagram is identical to Fig. 20; hence they represent the same knot. Moreover, the mirror image (lateral inversion, left to right) of each diagram is identical to itself. Therefore the mirror image produces the same knot again; we shall see later that this is not the case with the other two arrangements.

Recall (from page 15) that when $\Delta = 0$, Type 1 is equal to Type 2. The above analysis of Figs. 19 and 20 exemplifies this.

Herringbone Pineapple Knots which have their left- and right-hand nests arranged as in these Figures belong either to the Standard Herringbone Pineapple Knots or the Semi-Standard Herringbone Pineapple Knots. Neither of these can be braided from a single string, but have to be braided from multiple strings. For more details the reader can refer to Appendix 2.

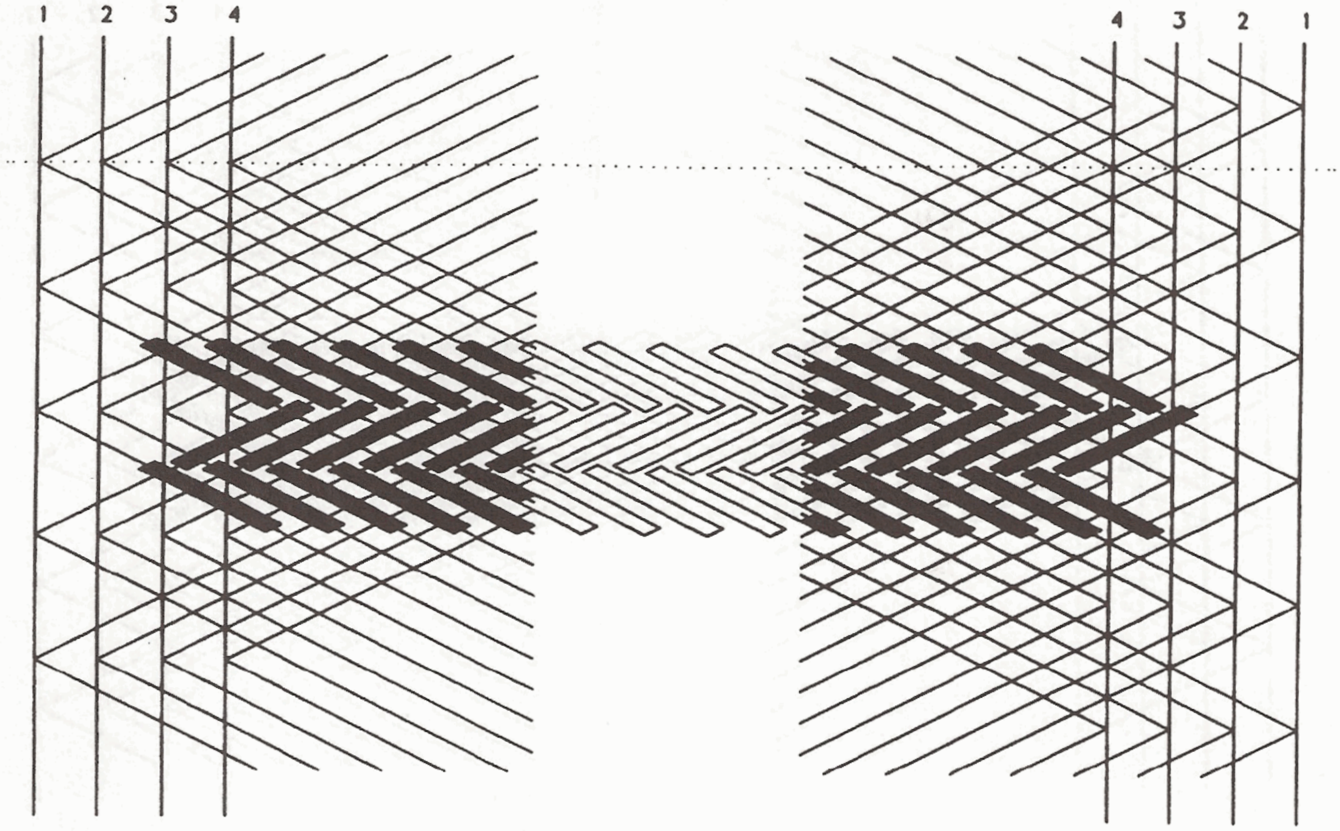
The second and third possible arrangements referred to above are represented respectively in Figs. 21 and 22. For the second arrangement we have a Δ value that is equal to 1 (see Fig. 21). For the third arrangement, we have a Δ value which is equal to $(A-1)$ (see Fig. 22).

Note that $(A-1) = |-1|_A$ and hence $\Delta = A - 1$ is equivalent to $\Delta = -1$.

The reader should now observe that by turning Fig. 21 through an angle of 180° about an axis perpendicular to the drawing plane, we obtain a Figure identical to the original one. Similarly for Fig. 22. Then, since the coding arrangements of Fig. 21 and Fig. 22 are evidently different from one another, it is clear that the two Figures represent different knots. Nevertheless, Figs. 21 and 22 may be seen to be mirror images of one another. Hence the mirror image of either one Figure represents the same knot as that knot represented by the other Figure.

Herringbone Pineapple Knots which have their left- and right-hand nests arrranged as in one of the Figs. 21 or 22 belong to either the Perfect Herringbone Pineapple Knots or the Semi-Perfect Herringbone Pineapple Knots. The Perfect Herringbone Pineapple Knots are made from a single string, whereas Semi-Perfect Herringbone Pineapple Knots require more than one string. These Herringbone Pineapple Knots will be fully discussed in a later volume of our Series of Books on Braiding.

The value of Δ is involved in expressing the relative positions of the nests in rows on the left- and right-hand edges of the Regular-Nested Cylindrical Braid. Another example to illustrate this is given in Fig. 23.



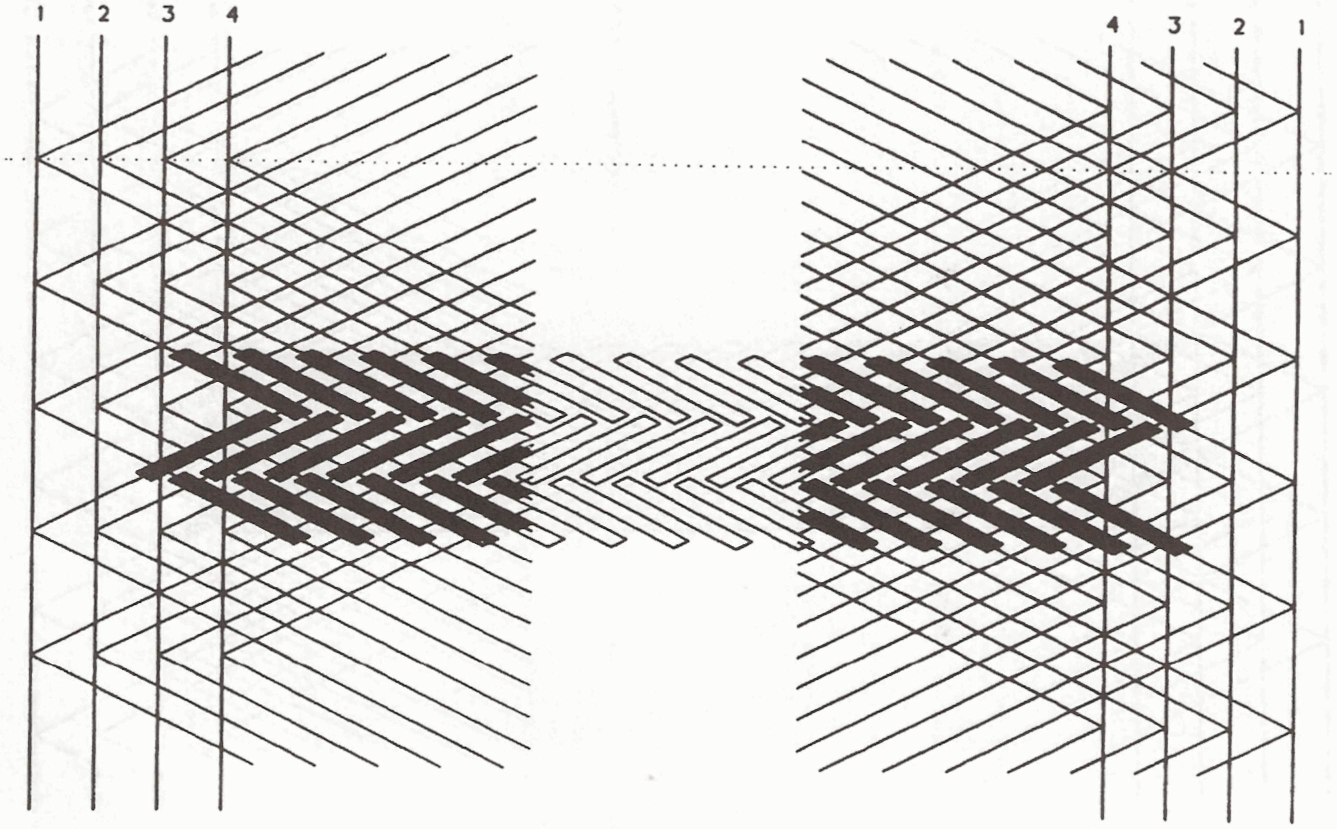
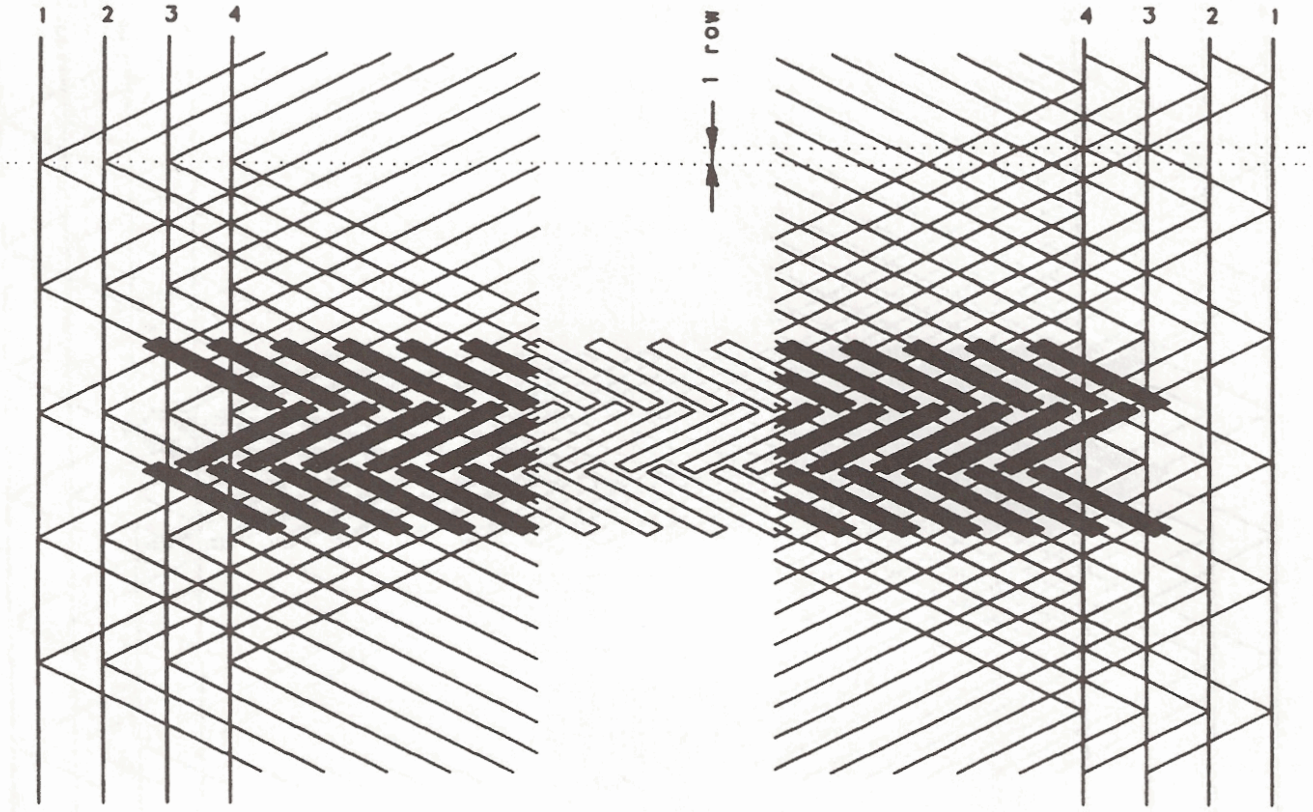


Fig. 19 (top) — The herringbone coding arrangements
Fig. 20 (bottom) of a herringbone coded Pineapple Knot with $\Delta = 0$.



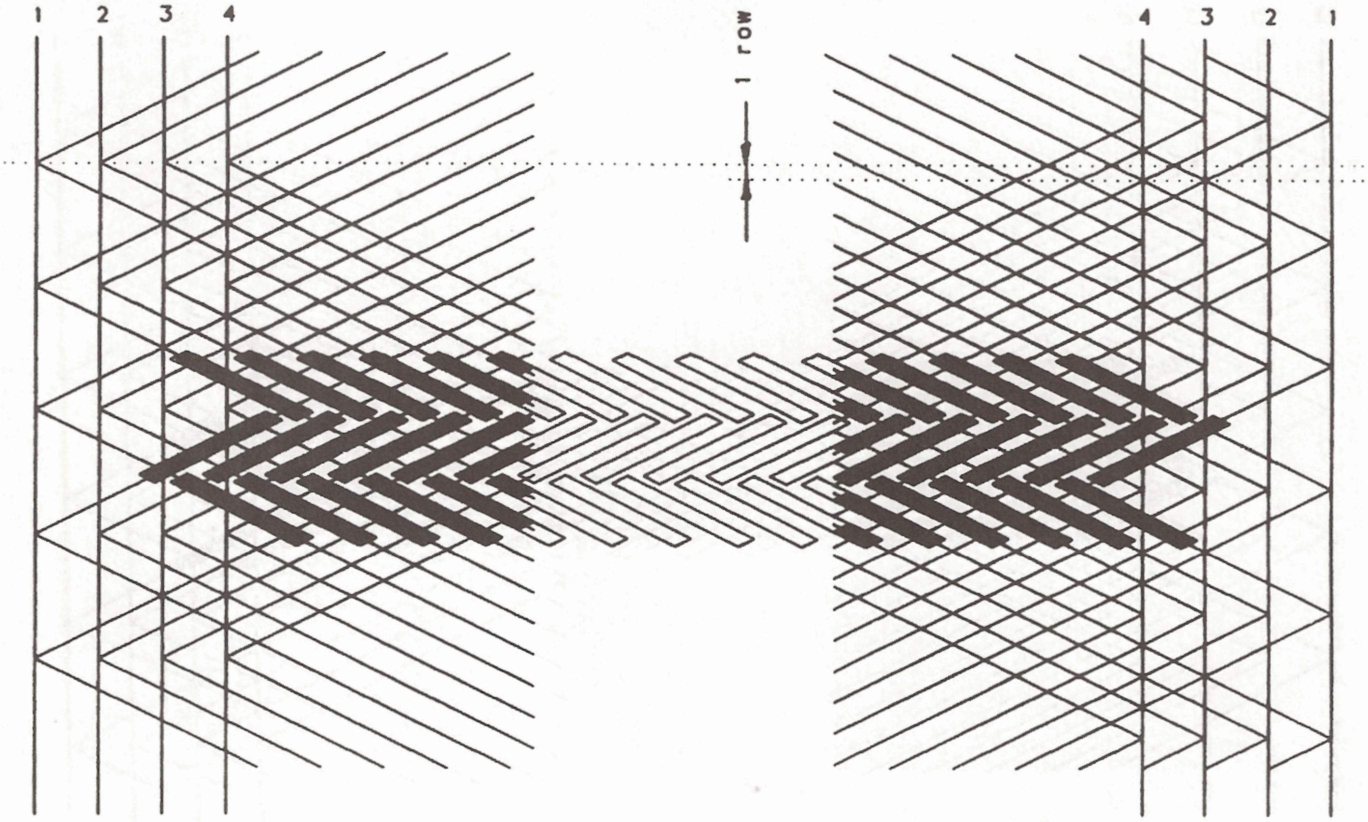
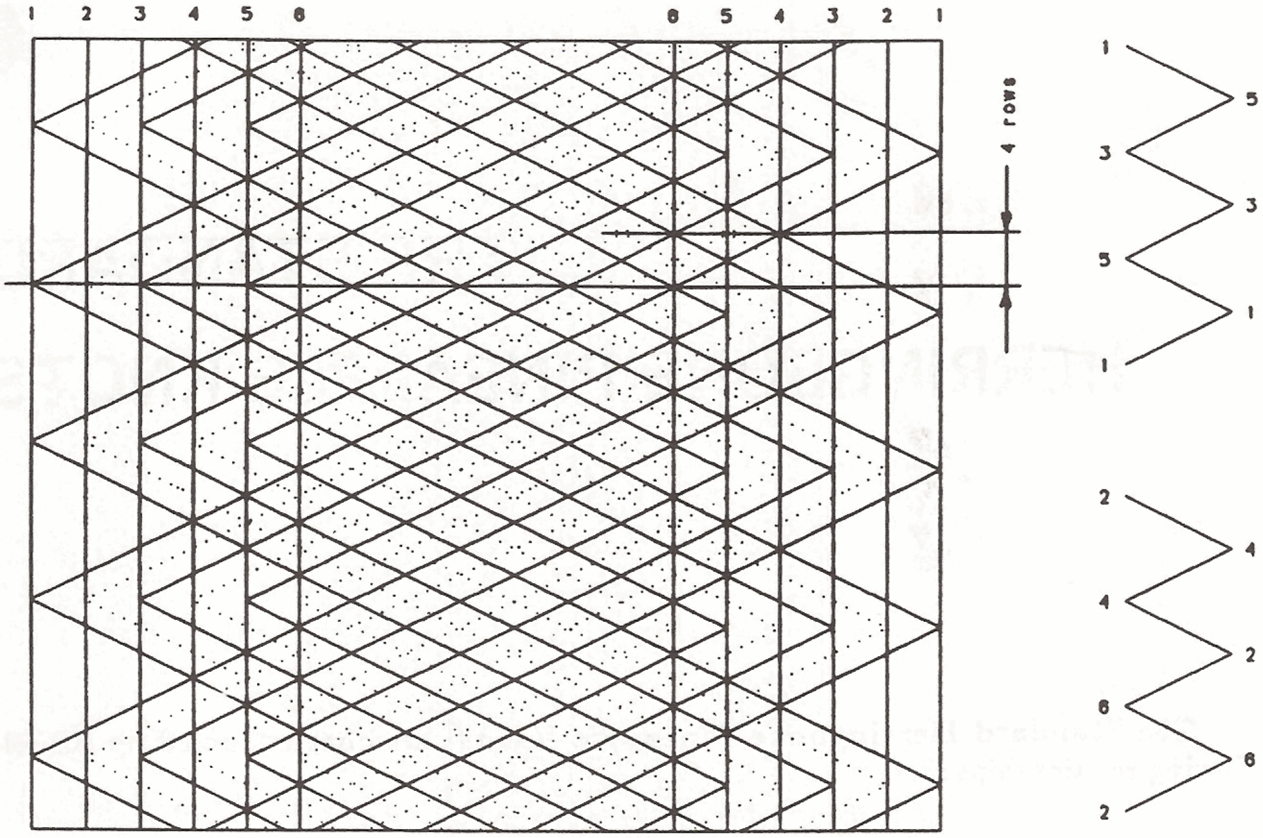


Fig. 21 (top) — The herringbone coding arrangements of a herringbone coded
Fig. 22 (bottom) Pineapple Knot with $\Delta = 1$ respectively $\Delta = A - 1$.



A=6; Δ=4; B*=5; P=24; x=14; P_{comp}=13(1 off); P_{comp}=11(1 off)

Fig. 23 — The string-run of a Compound Regular-Nested Cylindrical Braid.

The reader will find it instructive to do the following, with regard to Fig. 23:

- Check that A (the number of bight-boundaries per braid-edge) is 6.
- Check that the number of nests per braid-edge is $B^* = 5$.
- Check that x (the number of columns between the left- and right- bight-boundaries A) is 14.
- Discover the two first-return string-runs, as shown at the right of the Figure.
- Note that the whole braid is made with two strings. Each string contributes a component to the total braid.
- Check that one component has 13 parts, and that the other has 11 parts.
- Check that $\Delta = 4$, by comparing l_i and l_{i+1} in a first-return string-run, of both components.

It should be realised that when designing a knot, the four quantities A, Δ, B^* and x are decided upon first; and then, from these the related grid-diagram is drawn. Restrictions and relationships on these quantities will determine the kind of knot that will result. Appendices 1 and 2 deal with these conditions and relationships.

4

THE STANDARD HERRINGBONE PINEAPPLE KNOTS

The **Standard Herringbone Pineapple Knots**† **are characterized** by the following relationships:

$$\Delta = 0 \tag{1}$$

$$B^* \text{ and } P_{component} \text{ are coprime} \tag{2}$$

$$x = cA + \delta, \qquad \text{wherein} \tag{3}$$

$$\delta = |2(l_i + r_i)|_A \qquad \text{and} \tag{4}$$

$$c = (2m - 3) + \frac{2(l_i + r_i) - \delta}{A} \tag{5}$$

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{A} \tag{6}$$

$$P = \sum P_{component} = x + 2A - 2 \tag{7}$$

If and only if the conditions (1), (2) and (5) are all met, then a Standard Herringbone Pineapple Knot will result. They are necessary and sufficient conditions for this type of Knot.

The following remarks on each condition are instructive:

(1) This condition is a requirement for all Standard- and Semi-Standard Regular-Nested Cylindrical Braids (we attach the suffix 'Semi' when a component cannot be made from one string).

(2) This condition ensures that each component can be made from a single string.

---

† *The Standard Herringbone Pineapple Knots are generally referred to in the literature as Pineapple Knots; however this term is much too loose.*

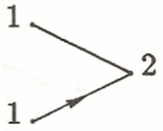
(3) This is true for *every* Regular-Nested Cylindrical Braid.
(4) This is true when $\Delta = 0$.
(5) This is a requirement for every Standard and Semi-Standard Herringbone Pineapple Knot.
(6) (number of parts per component) The formula applies to all Standard- and Semi-Standard Regular-Nested Cylindrical Braids.
(7) This formula, for 'total number of parts in complete knot', applies to every Regular-Nested Cylindrical Braid.

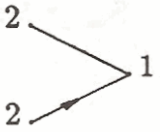
Example : *(see Figs. 24, 25, and 26, 27)*

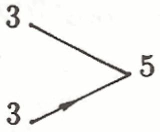
Suppose we have a Standard Herringbone Pineapple Knot, with $A = 5$ and a half-cycle running from lower-left to upper-right between the left bight-boundary $l_i = 3$ and the right bight-boundary $r_i = 5$. From condition (1) above, we know that $\Delta = 0$, hence $l_i = l_{i+1}$ and $r_i = r_{i+1}$.

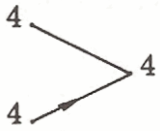
And since $l_i + r_i = 3 + 5 = 8$ in the given half-cycle, and this value is not equal to $A+1 = 6$, we conclude there are two sets of half-cycles, one for which $l_i + r_i = 8 (= Y + A$ and hence $Y = 3)$, and the other for which $l_i + r_i = Y = 3$.

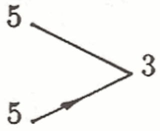
Using the methods of Chapter 2, on first-return string-runs (e.g. see Example 2 in that Chapter), we obtain the following first-return string-runs for this knot :











Hence we have five interwoven components with string runs as given above.

$$\delta = |2(l_i + r_i)|_A = |2(3+5)|_5 = |16|_5 = 1 .$$

$$\text{Hence} \quad c = (2m^* - 3) + \frac{(2 \times 8) - 1}{5} = 2m^* , \quad \text{for } l_i + r_i = 8 ;$$

$$\text{also} \quad c = (2m - 3) + \frac{(2 \times 3) - 1}{5} = 2m - 2 , \quad \text{for } l_i + r_i = 3 .$$

Since c is invariant we have $2m^* = 2m - 2$, thus $m^* = m - 1$.

$$\text{Now,} \quad x = cA + \delta = 10m - 9 , \quad \text{since } c = 2m - 2 , \; A = 5 , \; \delta = 1 .$$

So the values for $P_{component}$ are :

$$(i) \qquad P_{component} = \frac{x + 4A - 2(l_i + r_i)}{A}$$

$$= \frac{10m - 9 + (4 \times 5) - (2 \times 8)}{5} = 2m - 1 , \qquad \text{for} \quad l_i + r_i = 8 ;$$

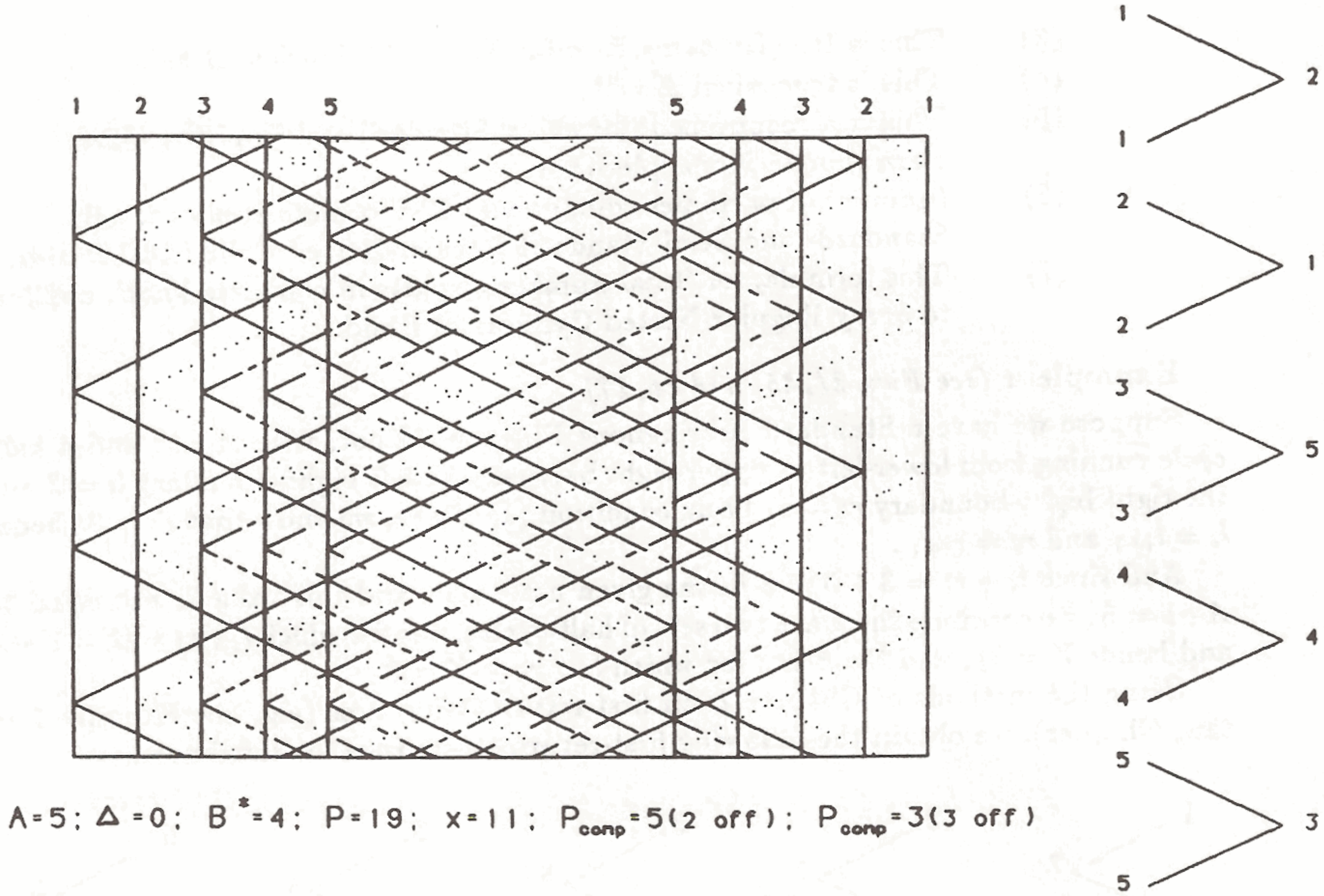
and

$$(ii) \qquad P_{component} = \frac{x + 4A - 2(l_i + r_i)}{A}$$

$$= \frac{10m - 9 + (4 \times 5) - (2 \times 3)}{5} = 2m + 1 , \qquad \text{for} \quad l_i + r_i = 3 .$$

Finally,

$$P_{total} = P = x + 2A - 2 = 10m - 9 + (2 \times 5) - 2 = 10m - 1 .$$


1 2 3 4 5 5 4 3 2 1
A=5; Δ=0; B*=4; P=19; x=11; P_conp=5(2 off); P_conp=3(3 off)
1 2 1
2 1 2
3 5 3
4 4 4
5 3 5

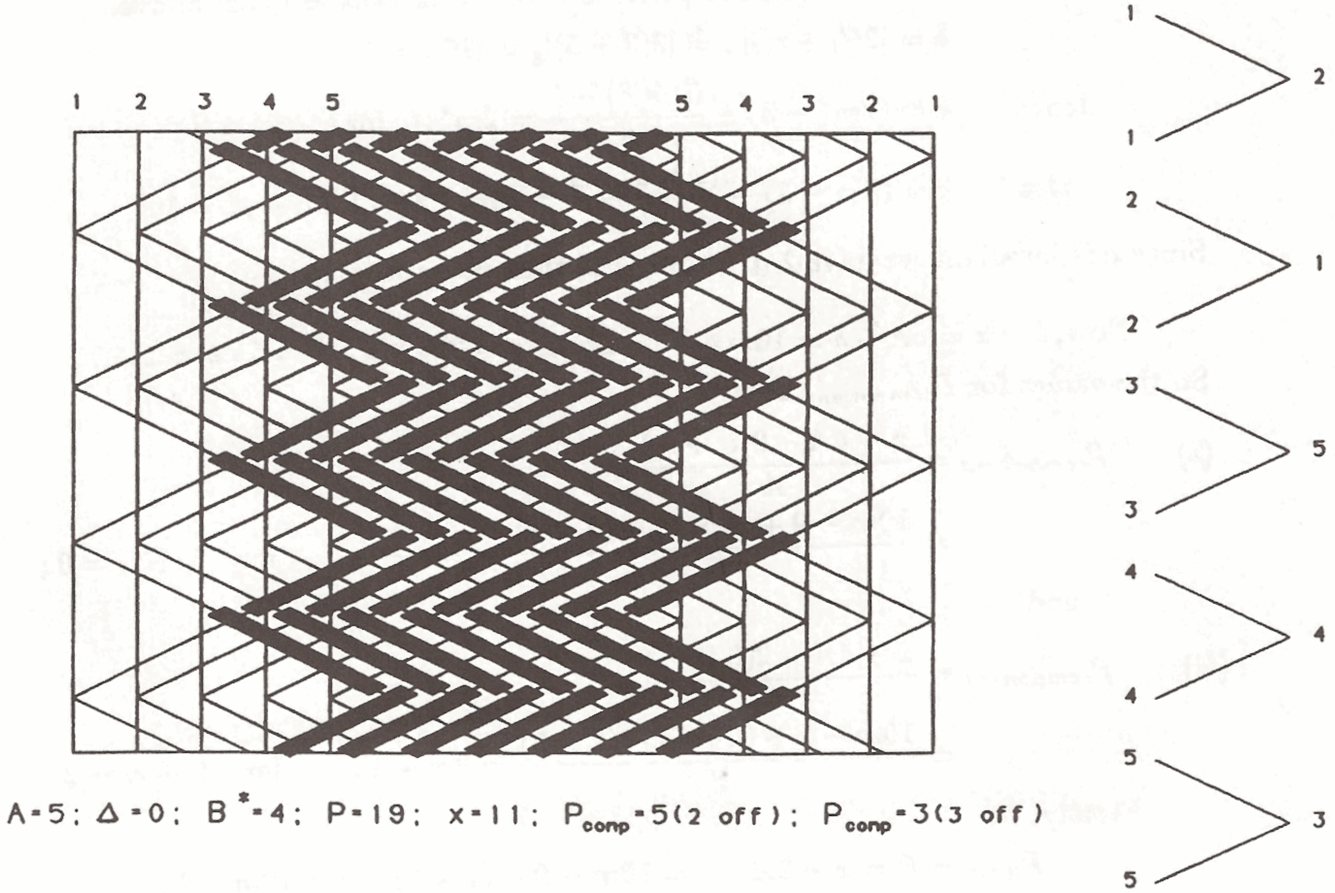
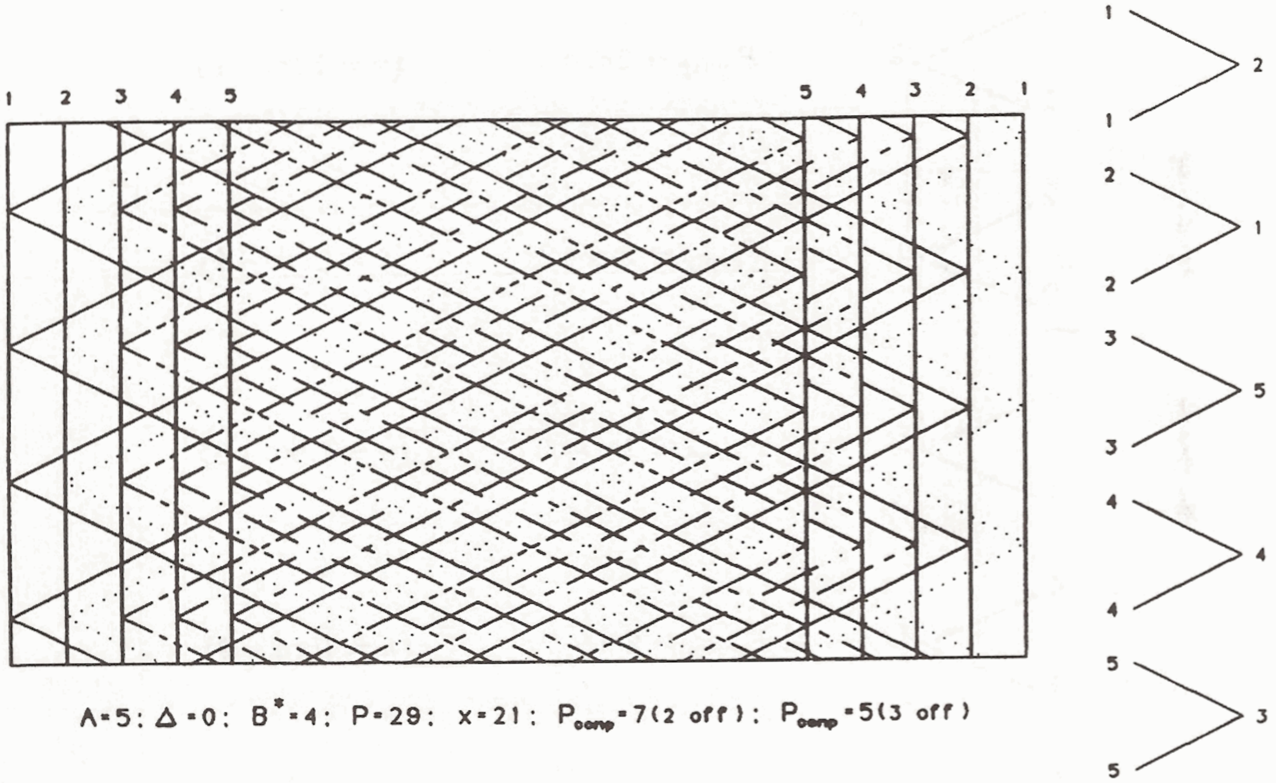
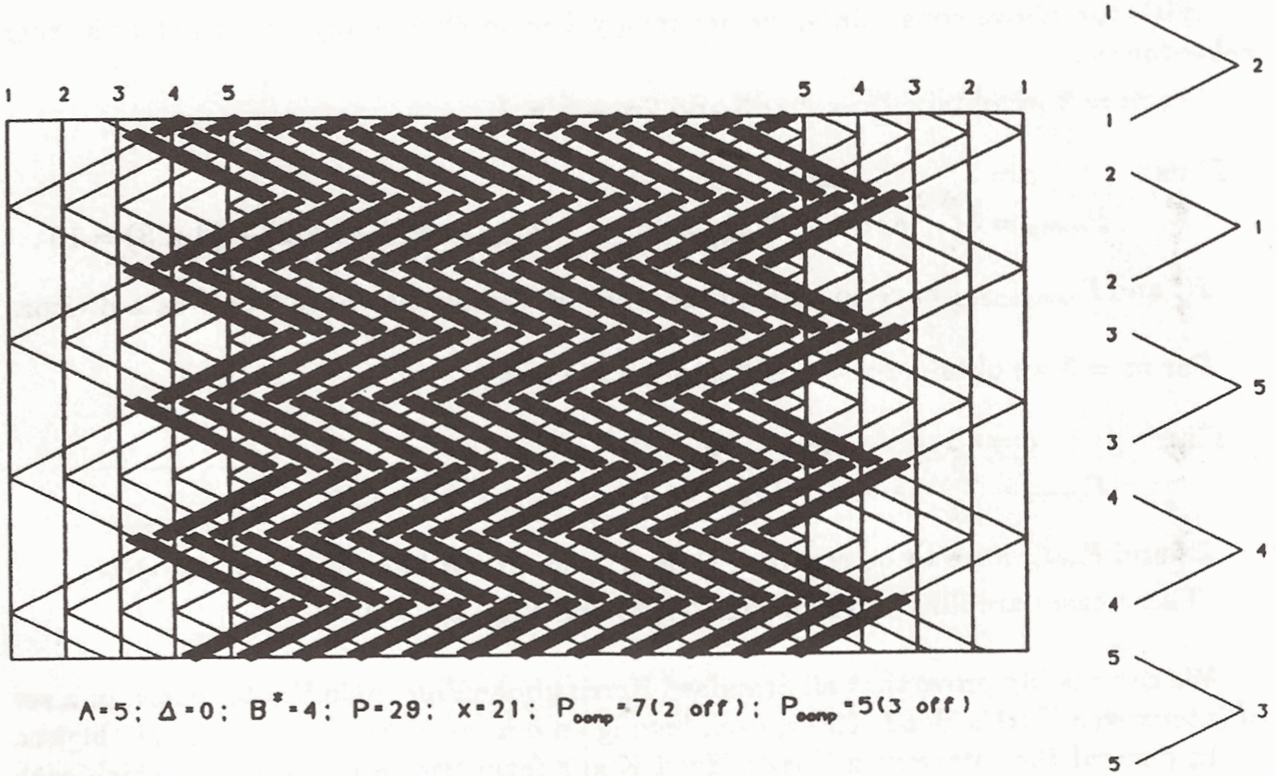

1 2 3 4 5 5 4 3 2 1
A=5; Δ=0; B*=4; P=19; x=11; P_conp=5(2 off); P_conp=3(3 off)
1 2 1
2 1 2
3 5 3
4 4 4
5 3 5

Fig. 24 (top)
Fig. 25 (bottom) — A Standard Herringbone Pineapple Knot



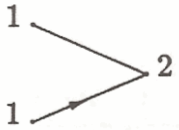
A=5; Δ=0; $B^*=4$; P=29; x=21; $P_{comp}=7$(2 off); $P_{comp}=5$(3 off)



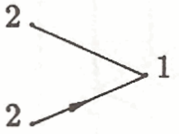
A=5; Δ=0; $B^*=4$; P=29; x=21; $P_{comp}=7$(2 off); $P_{comp}=5$(3 off)

Fig. 26 (top)
Fig. 27 (bottom) — A Standard Herringbone Pineapple Knot

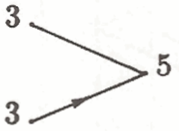
The first-return string-runs determining the five components are therefore as follows:



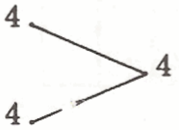
$P_{comp} = 2m + 1$ $(x = 10m - 9)$



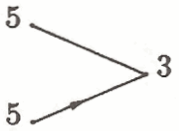
$P_{comp} = 2m + 1$ $(x = 10m - 9)$



$P_{comp} = 2m - 1$ $(x = 10m - 9)$



$P_{comp} = 2m - 1$ $(x = 10m - 9)$



$P_{comp} = 2m - 1$ $(x = 10m - 9)$

With the above constraints, we are finally free to choose any non-negative integer value for m .

For $m = 2$ we obtain $2m + 1 = 5$ and $2m - 1 = 3$.

Thus $c = 2$ and $x = 11$

$P_{comp} = 5$ and $P_{comp} = 3$; $P_{total} = P = (5 \times 2) + (3 \times 3) = 19$.

B^* and $P_{component}$ have to be coprime, thus B^* should not have 3 or 5 as a divisor.

For $m = 3$ we obtain $2m + 1 = 7$ and $2m - 1 = 5$.

Thus $c = 4$ and $x = 21$

$P_{comp} = 7$ and $P_{comp} = 5$; $P_{total} = P = 29$.

B^* and P_{comp} have to be coprime, thus B^* should not have 5 or 7 as a divisor. (These cases are illustrated in Figs. 24, 25 and Figs. 26, 27.)

We can readily prove that all Standard Herringbone Pineapple Knots consist of a set of interwoven Turk's Head Knots, each having an odd number of parts, and B^* bights.

In general the interwoven Turk's Head Knots form two sets, one set in which each knot (component) has a number of parts equal to $(2m - 1)$, and the other set in which each knot (component) has $(2m + 1)$ parts.

There is a special case wherein all the interwoven Turk's Head Knots have the same odd number of parts. This occurs when the Standard Herringbone Pineapple Knot has a half-cycle running from the left bight-boundary 1 to the right bight-boundary A .

It is often stated in books on knots and braiding that a (herringbone-coded) Pineapple Knot is formed by interbraiding Turk's Head Knots which have an odd number of parts. This is, however, not a necessity; we can form herringbone-coded Pineapple Knots by interbraiding Turk's Head Knots which have an even number of parts, although these Pineapple knots do not belong to the sub-family of the Standard Herringbone Pineapple Knots, but instead belong to the sub-family of the Semi-Perfect Herringbone Pineapple knots. We will treat these in a later volume.

5

THE ALGORITHM-TABLES FOR THE STANDARD HERRINGBONE PINEAPPLE KNOTS

Layout of Algorithm-tables

Before studying this chapter, the reader should be thoroughly familiar with the contents of pages 6 to 11 of Chapter 2, in order to have acquired a good understanding of the layout of the grid-diagrams of Regular-Nested Cylindrical Braids. It is advisable, but not essential, to study the other material which precedes this Chapter.

Any Standard Herringbone Pineapple Knot may be constructed by means of algorithms obtained from their universal Algorithm-tables. In order to understand how these Algorithm-tables work, it is necessary to study their general layout.

Any Standard Herringbone Pineapple Knot is produced by the interbraiding of Turk's Head Knots, all having the same number of bights, in a specific way. These Turk's Head Knots must have an **odd** number of parts; and they either have all the same odd number of parts, or else they form two sets. The members within a set all have the same odd number of parts. Thus there are two different odd part numbers, one for each set. These, two, odd part numbers in the respective sets differ by 2. Hence, either each of the interbraided Turk's Head Knots has $(2m-1)$ parts, or else, the interbraided Turk's Head Knots divide into two sets, such that each member of one set has $(2m-1)$ parts and each member of the other set has $(2m+1)$ parts.

We shall describe the general layout of the Algorithm-tables with the aid of the table in Fig. 28. The number of parts of an interbraided Turk's Head Knot is indicated by the value of the letter s printed above each table.

The crossings along a half-cycle in a completed Standard Herringbone Pineapple Knot can be divided into **sets** of crossings, whereby the crossings of a set are all adjacent and possess the same coding.

The number of sets of crossings associated with the interbraiding of a Turk's Head Knot is always equal to the number of parts of this interbraided Turk's Head Knot, hence equal to the value of s.

s=s

set number	**1**	**2**	**3**	**4**	…………	**s - 3**	**s - 2**	**s - 1**	**s**
crossing type	**u**	**o**	**u**	**o**	…………	**o**	**u**	**o**	**u**
ref. L → R	**L - 1**	**A - 1**	**A - 1**	**A - 1**	…………	**A - 1**	**A - 1**	**A - 1**	**R - 1**
ref. R → L	**R - 1**								**L - 1**
1. L → R	L - 1	A - 1	A - 1	A - 1	…………	A - 1	A - 1	A - 1	R - 1
2. R → L									L - 1
3. L → R									R - 1
4. R → L									L - 1
5. L → R									R - 1
⋮ ⋮									⋮
2B*-4. R → L									L - 1
2B*-3. L → R									R - 1
2B*-2. R → L									L - 1
2B*-1. L → R									R - 1
2B*. R → L									L - 1

Fig. 28 — General layout of the Algorithm-tables.

The first row in the tables gives the various set numbers associated with the Turk's Head Knot in question, and in the second row the coding of the crossings belonging to each set is indicated immediately below the appropriate set.

The third row in the tables consists in fact of two rows: the upper one, referring to a lower-left to upper-right half-cycle, and the lower one, referring to a lower-right to upper-left half-cycle. Here 'L' is the numerical-value of the left-hand bight-boundary and 'R' is the numerical value of the right-hand bight-boundary of the respective half-cycle. The reference values '$L-1$', '$R-1$' and '$A-1$' can be calculated by using the appropriate values for 'L', 'R' and 'A' associated with the Standard Herringbone Pineapple Knot which results after the Turk's Head Knot under consideration has been interbraided. Note that the entries '$A-1$' apply to both half-cycles.

The braiding algorithm for the first half-cycle is always as indicated in the general table for this first half-cycle:

$$(L-1)u - (A-1)o - (A-1)u - \cdots - (A-1)u - (A-1)o - (R-1)u\,.$$

The general table entries for the remaining half-cycles depend on the values of s and B^*. Note however that the last entry for each half-cycle depends on the values of 'L' and 'R' only.

Parameter formulae

From the above, the reader will have observed that before we can start with the construction of a Standard Herringbone Pineapple Knot, we have to know the values for the parts of the interbraided Turk's Head Knots (the values for s). (The value of B^* will in practice have been chosen by the braider from the outset.)

Let us assume that the values for A and $P_{total} = P$ are given, and let us further assume that the Standard Herringbone Pineapple Knot consists of a_1 interbraided Turk's Head Knots, each having $(2m-1)$ parts, and a_2 interbraided Turk's Head Knots, each having $(2m+1)$ parts, where m is a positive integer.

It will be evident that one of these two sets of interbraided Turk's Head Knots may be empty, or, in other words, a_1 or a_2 may be zero; in such case we can always choose a_2 to be zero.

Thus we obtain:

$$A = a_1 + a_2 \,,$$
$$P = a_1(2m-1) + a_2(2m+1) = 2m \cdot A + (a_2 - a_1) \,.$$

When A is **odd**, then a_1 and a_2 are of opposite parity, and hence P is **odd**.

When A is **even**, then a_1 and a_2 are of the same parity, and hence P is even.

The relationship $P = 2m \cdot A + (a_2 - a_1)$ gives us:

$$m = \frac{P - (a_2 - a_1)}{2A} \,.$$

Furthermore $|P|_A = |a_2 - a_1|_A \,.$

Thus $|P|_A = a_2 - a_1$ when $1 \le a_1 \le a_2$, or
$|P|_A = A + (a_2 - a_1)$ when $a_1 > a_2 \ge 0$.

Since $a_2 = A - a_1$ we can write :
$|P|_A = A - 2a_1$ when $1 \le a_1 \le a_2$, or
$|P|_A = 2A - 2a_1$ when $a_1 > a_2 \ge 0$.

Hence

$$\left.\begin{aligned} a_1 &= \frac{A - |P|_A}{2} \\ a_2 = A - a_1 &= \frac{A + |P|_A}{2} \\ 2m &= \frac{P - |P|_A}{A} \end{aligned}\right\} \text{ when } 1 \le a_1 \le a_2 \,,$$

or

$$\left.\begin{aligned} a_1 &= \frac{2A - |P|_A}{2} \\ a_2 = A - a_1 &= \frac{|P|_A}{2} \\ 2m &= \frac{P - |P|_A}{A} + 1 \end{aligned}\right\} \text{ when } a_1 > a_2 \ge 0 \,.$$

For A and P both odd:

Since a_1 has to be an integer, it follows that:

For $|P|_A$ is odd: $a_1 = \dfrac{A - |P|_A}{2}$; $a_2 = \dfrac{A + |P|_A}{2}$ and $2m = \dfrac{P - |P|_A}{A}$.

For $|P|_A$ is even: $a_1 = \dfrac{2A - |P|_A}{2}$; $a_2 = \dfrac{|P|_A}{2}$ and $2m = \dfrac{P - |P|_A}{A} + 1$.

For A and P both even:

$$|P|_A = \text{even}$$

$$2m = \frac{P - (a_2 - a_1)}{A} = \text{even}$$

Hence:

$$a_1 = \frac{A - |P|_A}{2} \; ; \; a_2 = \frac{A + |P|_A}{2} \qquad \text{when } 2m = \frac{P - |P|_A}{A} = \text{even} ,$$

$$a_1 = \frac{2A - |P|_A}{2} \; ; \; a_2 = \frac{|P|_A}{2} \qquad \text{when } 2m = \frac{P - |P|_A}{A} + 1 = \text{even} .$$

The above formulae enable us to calculate the values of the important parameters of the interbraided Turk's Head Knots which make up a particular Standard Herringbone Pineapple Knot. These are the values for a_1 with $s = 2m-1$, and for a_2 with $s = 2m+1$ when a_2 is greater than zero.

Thus in general we will have to use two tables during the construction of a Standard Herringbone Pineapple Knot; one in which $s = 2m-1$ and the other in which $s = 2m+1$. For this reason we have printed (at the end of this Chapter) two consecutive tables on each of the table-pages, unless the value of B^* is such that there are no two consecutive s values which are both coprime with it. (Recall that s is the value of $P_{component}$, and that B^* must be coprime to this; see Chapter 4, page 28.)

Before we can use these tables, we have to know how the interbraided Turk's Head Knots are positioned in the Standard Herringbone Pineapple Knot under consideration. This positioning we can schematically illustrate in a diagram as given in Fig. 29, where, from each interbraided Turk's Head Knot, only a half-cycle from lower-left to upper-right is displayed.

In general we have a_2 Turk's Head Knots, each with $s = 2m + 1$ parts. Their lower-left to upper-right half-cycles start at the left bight-boundaries $1, 2, 3, \cdots, (a_2 - 1), a_2$ and run to the respective right bight-boundaries $a_2, (a_2 - 1), (a_2 - 3), \cdots, 2, 1$.

The a_1 lower-left to upper-right half-cycles of the Turk's Head Knots, each with $s = 2m-1$ parts, start at the left bight-boundaries $(a_2+1), (a_2+2), (a_2+3), \cdots, (A-1), A$ and run to the respective right bight-boundaries $A, (A-1), (A-2), \cdots, (a_2+2), (a_2+1)$.

When $a_2 = 0$ (hence the Standard Herringbone Pineapple Knot consists of $a_1 = A$ interbraided Turk's Head Knots, each with $s = 2m-1$ parts) the reader will observe that the half-cycles from lower-left to upper-right of these Turk's Head Knots start at the left bight-boundaries $1, 2, 3, \cdots, (A - 1), A$ and run to the respective right bight-boundaries $A, (A - 1), (A - 2), \cdots, 2, 1$. This is illustrated in Fig. 30.

It is important to note here that an A-pass Standard Herringbone Pineapple Knot can be braided in $A!$ different ways ($A!$ stands for the answer to the multiplication $1 \times 2 \times 3 \times 4 \times \cdots \times (A - 2) \times (A - 1) \times A$; for example $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$) .

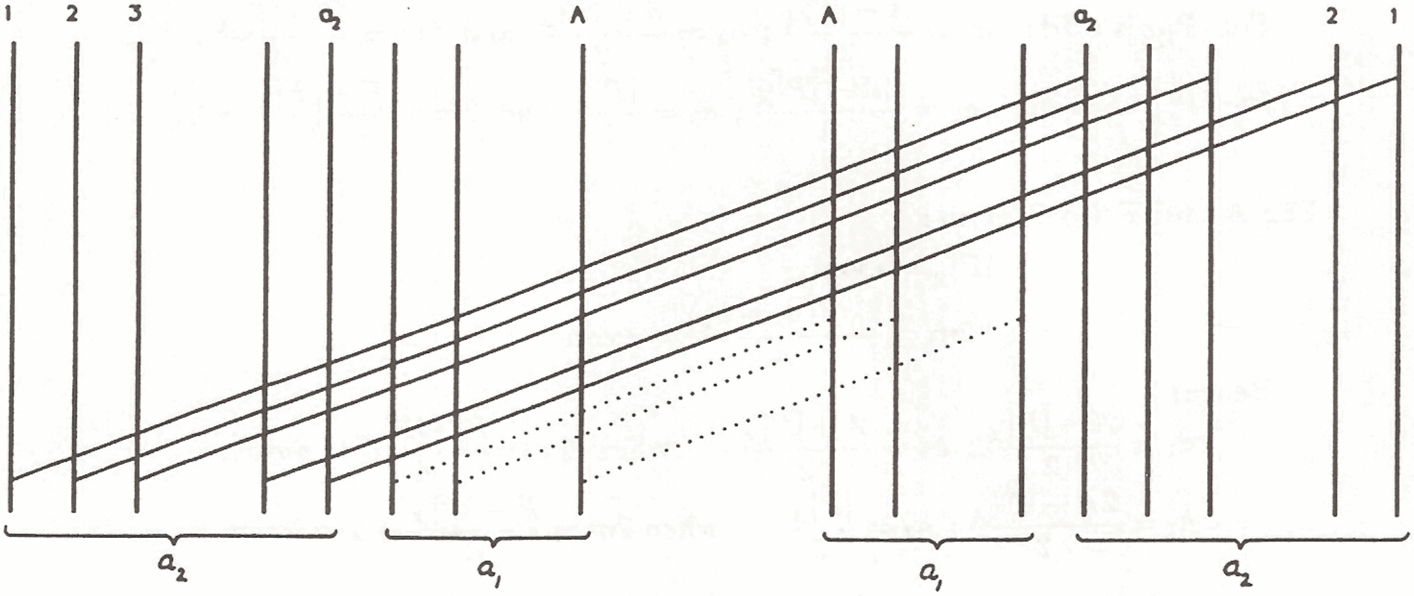


Fig. 29 — The set of a_1 half-cycles with $s = 2m - 1$, and the set of a_2 half-cycles with $s = 2m + 1$.

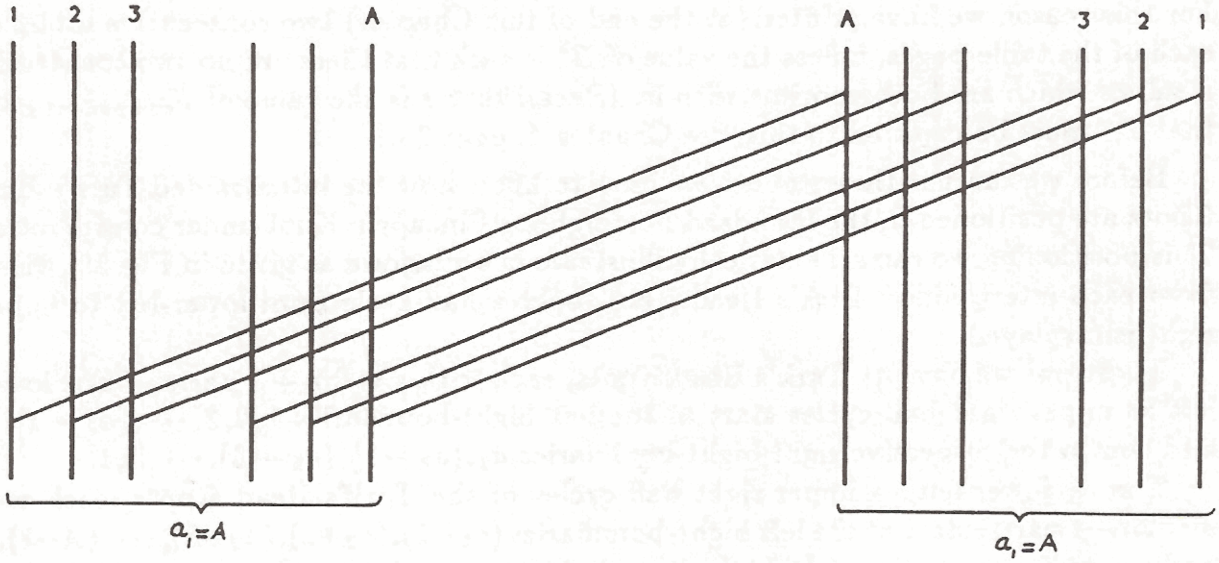


Fig. 30 — $a_2 = 0$ and hence $a_1 = A$ with $s = 2m - 1$.

After our discussion of the general table layout, the calculation of the interwoven Turk's Head Knot parameters and the schematic diagram illustrating their positioning, we are now able to show, step by step, the actual construction of a Standard Herringbone Pineapple Knot. We do this by means of example calculations, making use of Algorithm-tables which are given at the end of this Chapter.

Worked examples

Example 1:

Suppose that we want to braid a 2-Pass Standard Herringbone Pineapple Knot (hence $A = 2$), with $B^* = 4$ and $P = 12$.

Thus $$P = 12 = \mathbf{even}\,,$$
$$|P|_A = |12|_2 = 0\,,$$
$$2m = \frac{P - |P|_A}{A} = \frac{12 - 0}{2} = 6 = \mathbf{even}\,.$$

Hence $$a_1 = \frac{A - |P|_A}{2} = \frac{2 - 0}{2} = 1\,,$$
$$a_2 = \frac{A + |P|_A}{2} = \frac{2 + 0}{2} = 1\,.$$

Furthermore $2m - 1 = 6 - 1 = 5$ and $2m + 1 = 6 + 1 = 7$.

This Standard Herringbone Pineapple Knot consists of two sets of interbraided Turk's Head Knots; one set consists of 1 Turk's Head Knot with 5 parts and the other set consists of 1 Turk's Head Knot with 7 parts. The positioning is therefore as shown in Fig. 31.

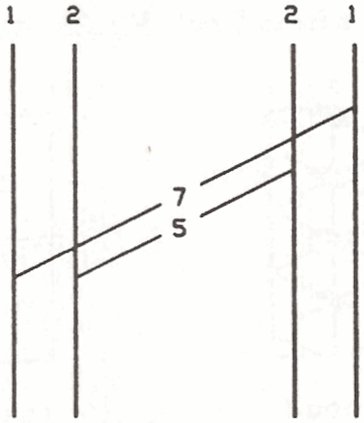
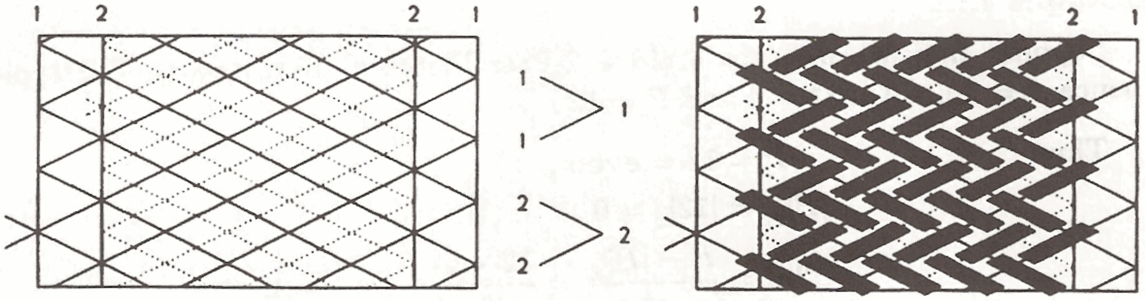


Fig. 31 — The positioning of the interbraided Turk's Head Knots.

Since $A = 2$ we can braid this Standard Herringbone Pineapple Knot in $2! = 2$ different ways. We shall discuss in detail these 2 different ways.

The grid-diagram (string-run only) of this Standard Herringbone Pineapple Knot together with the first-return string-runs of the two interbraided Turk's Head Knots (components) is given in the left diagram of Fig. 32.

Adjacent bight-boundaries are always two columns apart, and the value of x can readily be obtained from the relationship $P = x + 2A - 2$, which gives $x = P - 2A + 2$. Hence for our example we obtain $x = 12 - (2 \times 2) + 2 = 10$; this means that the distance between the left bight-boundary '2' and the right bight-boundary '2' is equal to 10 columns. The distance between the top horizontal line and the bottom horizontal line is equal to $A \cdot B^*$ bight-units (1 bight-unit is equal to 2 rows); hence in our example there are $2 \times 4 = 8$ bight-units. The grid-diagram (string-run and coding) of our example Standard Herringbone Pineapple Knot is given in the right diagram of Fig. 32.



$A=2$: $\Delta=0$: $B^{*}=4$: $P=12$: $x=10$: $P_{oonp}=7$(1 off): $P_{oonp}=5$(1 off)

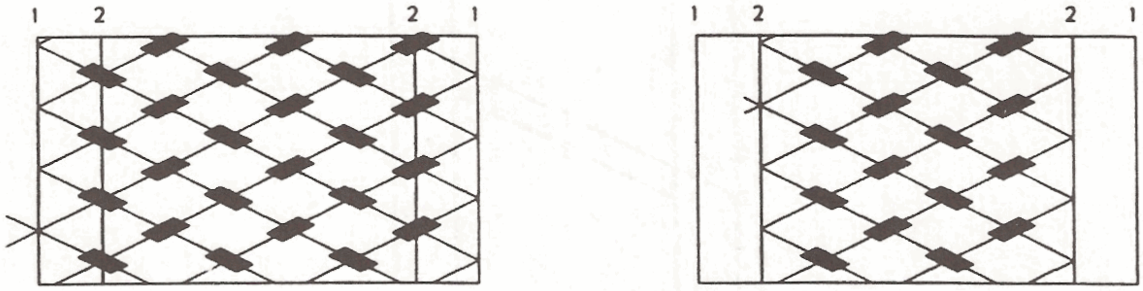
Fig. 32 — The Standard Herringbone Pineapple Knot of example 1.

It is recommended that the reader draws up some grid-diagrams of different Standard Herringbone Pineapple Knots on Isometric graph paper (see Ref. [1]).

(i) Let us assume that we want to braid the given Standard Herringbone Pineapple Knot in the following way:

Step 1. — Braid the component with half-cycle 1 ⟶ 1.
Step 2. — Braid the component with half-cycle 2 ⟶ 2.

The component in 'Step 1' is the foundation Turk's Head Knot and the component in 'Step 2' is the interwoven Turk's Head Knot. Fig. 33 gives their respective grid-diagrams.



Foundation Turk's head
7p/4b

Interwoven Turk's head
5p/4b

Fig. 33 — The interbraided Turk's Head Knots.

When 'Step 1' is completed we have braided a Turk's Head Knot with 4-bights and 7-parts. This knot, the foundation Turk's Head Knot, can also be regarded as a Standard Herringbone Pineapple Knot with $A = 1$ and $B^* = 4$. Since $A = 1$ the left bight-boundary as well as the right bight-boundary of this knot carries the number 1; hence in the table which will supply us with the braiding algorithm for this knot the value of 'L' is 1 and the value of 'R' is 1. Furthermore the value of 'A' is 1; the value of 'B^*' is 4 and the value of 's' is 7.

The position of a half-cycle from lower-left to upper-right belonging to this knot is schematically given together with the values for L, R and A in the left diagram of Fig. 34.

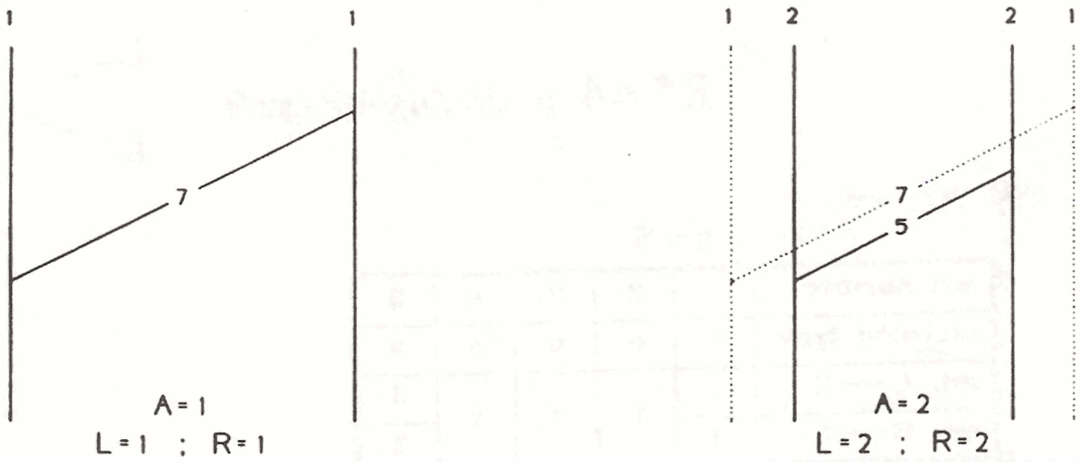


Fig. 34 — The position of a left to right half-cycle of the foundation and the interwoven Turk's Head Knots.

After substitution of $A = 1$, $L = 1$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 7$ we obtain $L - 1 = 0$, $R - 1 = 0$, $A - 1 = 0$ and hence the lower table of Fig. 35 results. This table supplies us with the following braiding algorithm associated with 'Step 1':

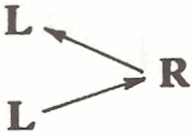
1. $L \longrightarrow R$: free run.
2. $R \longrightarrow L$: o.
3. $L \longrightarrow R$: o.
4. $R \longrightarrow L$: $u - o - u$.
5. $L \longrightarrow R$: $u - o - u$.
6. $R \longrightarrow L$: $u - 2o - u - o$.
7. $L \longrightarrow R$: $u - 2o - u - o$.
8. $R \longrightarrow L$: $u - o - u - o - u - o$.

The respective grid-diagrams associated with the above half-cycles are given in Fig. 36.

After completion of 'Step 1', we are ready to start with 'Step 2', which is the interbraiding of the Turk's Head Knot with $s = 5$ parts. When this interbraiding is completed, we have two left bight-boundaries and two right bight-boundaries; hence $A = 2$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1 and 2, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 1, and a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 2. Hence for the 'Step 2' Turk's Head Knot the value for 'L' is 2 and the value for 'R' is 2, with $A = 2$ and $s = 5$.

The positions of the half-cycles from lower-left to upper-right belonging to the 'Step 1' and 'Step 2' Turk's Head Knots are schematically illustrated, together with the values for L, R and A, in the right diagram of Fig. 34.

$B^*=4$; *Enlarging cycle*



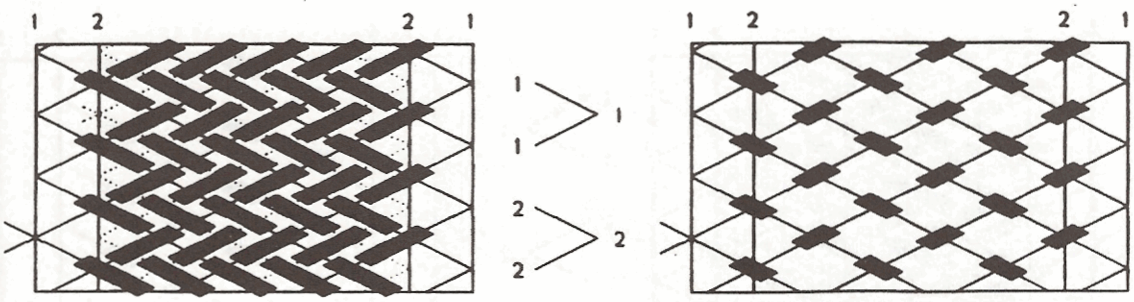
$s=5$

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	1	1	1	1	1
ref. R → L	1				1
1. L → R	1	1	1	1	1
2. R → L	1	1	1	2	1
3. L → R	1	1	1	2	1
4. R → L	1	1	2	2	1
5. L → R	1	1	2	2	1
6. R → L	1	2	2	2	1
7. L → R	1	2	2	2	1
8. R → L	2	2	2	2	1

$s=7$

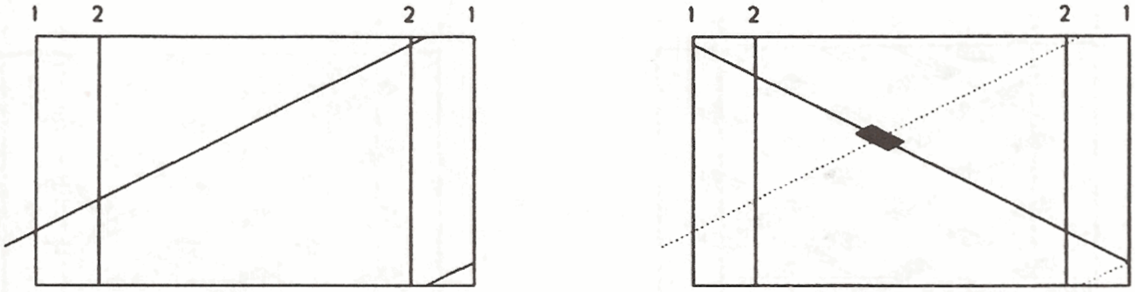
set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	0	0	0	0	0	0	0
ref. R → L	0						0
1. L → R	0	0	0	0	0	0	0
2. R → L	0	0	0	1	0	0	0
3. L → R	0	0	0	1	0	0	0
4. R → L	1	0	0	1	1	0	0
5. L → R	1	0	0	1	1	0	0
6. R → L	1	1	0	1	1	1	0
7. L → R	1	1	0	1	1	1	0
8. R → L	1	1	1	1	1	1	0

Fig. 35 — The resulting Algorithm-tables.



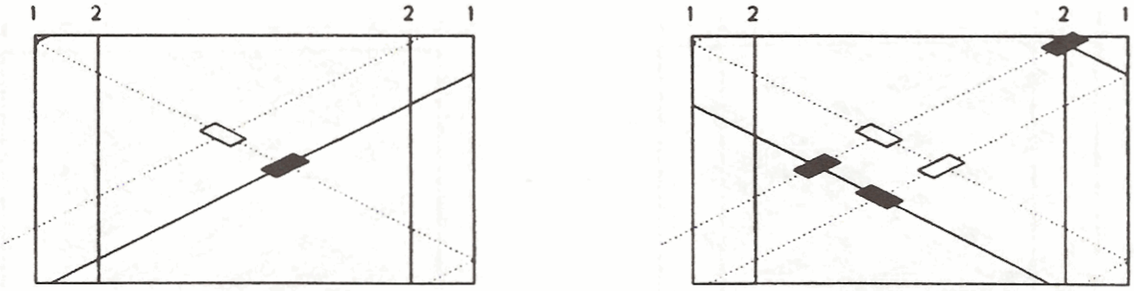
A=2; Δ=0; B* =4; P=12; x=10

Foundation Turk's head 7p/4b



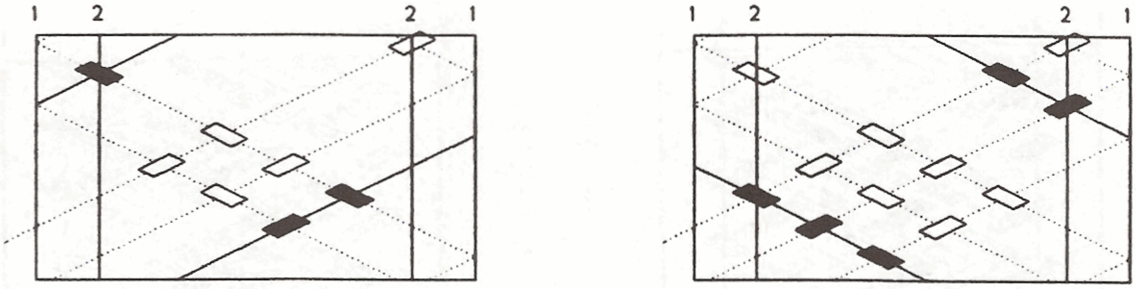
Half-cycle 1

Half-cycle 2



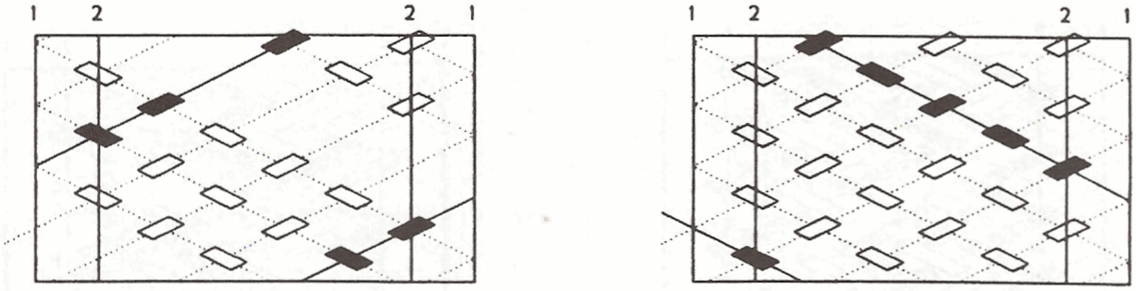
Half-cycle 3

Half-cycle 4



Half-cycle 5

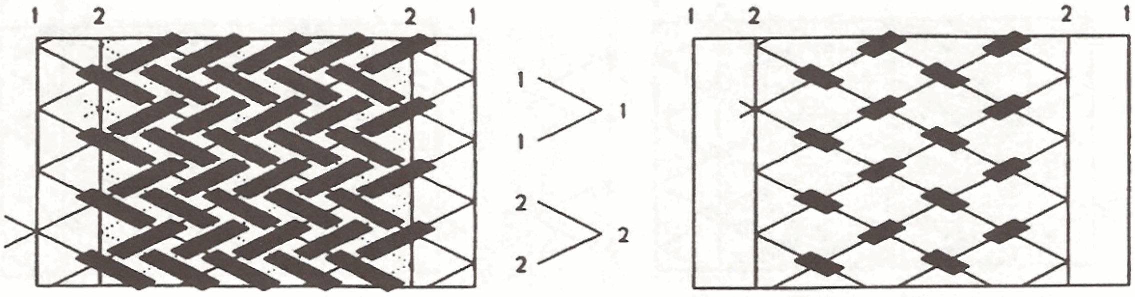
Half-cycle 6



Half-cycle 7

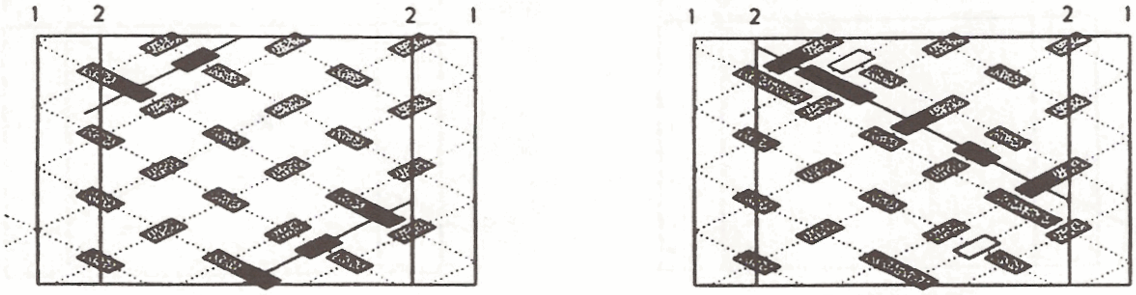
Half-cycle 8

Fig. 36 — The half-cycles in the braiding of the foundation Turk's Head Knot.



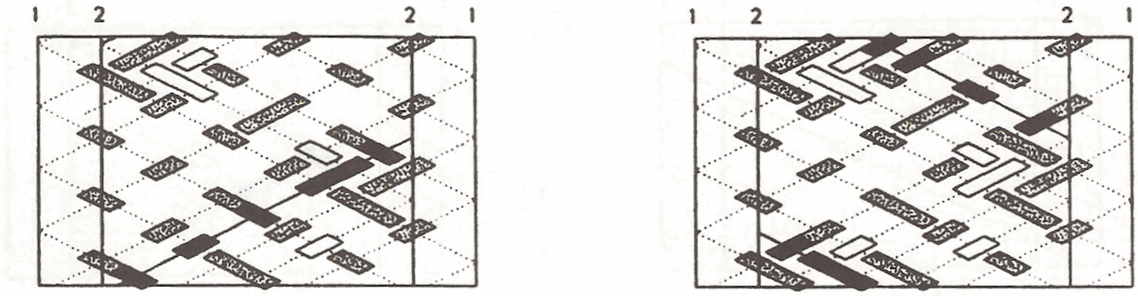
A=2; Δ=0; B^*=4; P=12; x=10

Interwoven Turk's head 5p/4b



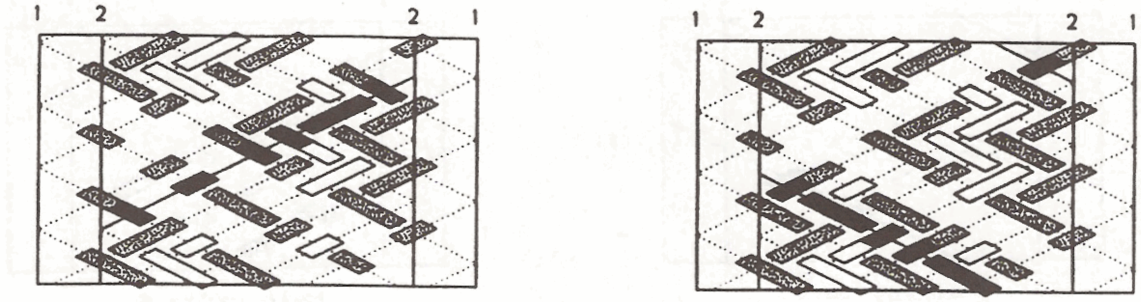
Half-cycle 1

Half-cycle 2



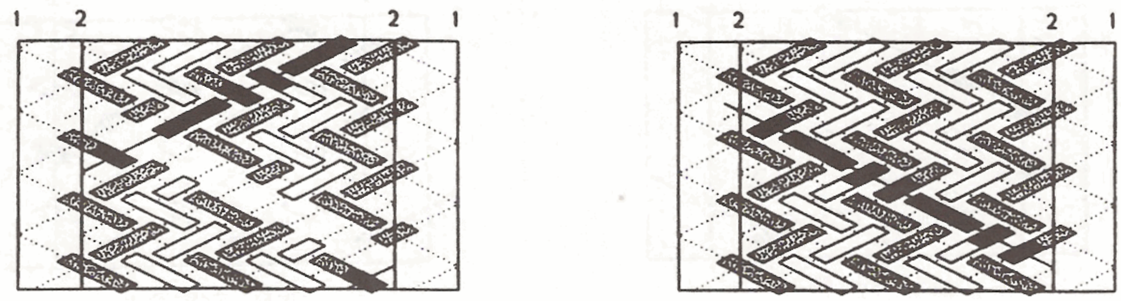
Half-cycle 3

Half-cycle 4



Half-cycle 5

Half-cycle 6



Half-cycle 7

Half-cycle 8

Fig. 37 — The half-cycles in the braiding of the interwoven Turk's Head Knot.

After substitution of $A = 2$, $L = 2$, $R = 2$ in the Algorithm-table for $B^* = 4$, $s = 5$ we obtain $L - 1 = 1$, $R - 1 = 1$, $A - 1 = 1$ and hence the upper table of Fig. 35 results. This table supplies us with the following braiding algorithm associated with 'Step 2':

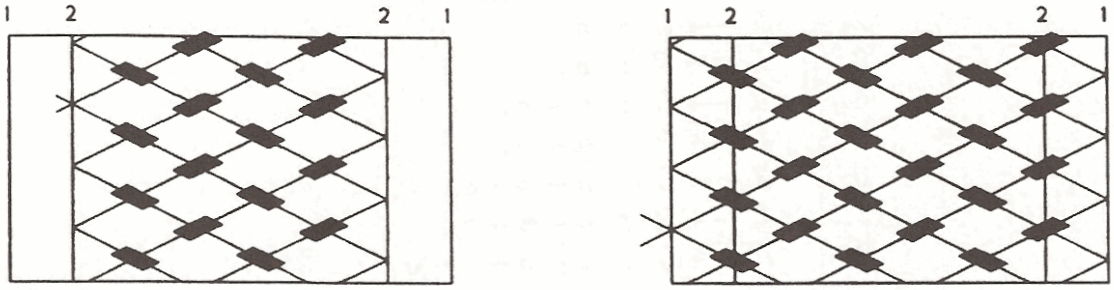
1. $L \longrightarrow R$: $u - o - u - o - u$.
2. $R \longrightarrow L$: $u - o - u - 2o - u$.
3. $L \longrightarrow R$: $u - o - u - 2o - u$.
4. $R \longrightarrow L$: $u - o - 2u - 2o - u$.
5. $L \longrightarrow R$: $u - o - 2u - 2o - u$.
6. $R \longrightarrow L$: $u - 2o - 2u - 2o - u$.
7. $L \longrightarrow R$: $u - 2o - 2u - 2o - u$.
8. $R \longrightarrow L$: $2u - 2o - 2u - 2o - u$.

The respective grid-diagrams associated with the above half-cycles are given in Fig. 37.

(ii) Let us now assume that we want to braid the given Standard Herringbone Pineapple Knot in the following way:

Step 1. — Braid the component with half-cycle $2 \longrightarrow 2$.
Step 2. — Braid the component with half-cycle $1 \longrightarrow 1$.

The component in 'Step 1' is the foundation Turk's Head Knot and the component in 'Step 2' is the interwoven Turk's Head Knot. Fig. 38 gives their respective grid-diagrams.



Foundation Turk's head
5p/4b

Interwoven Turk's head
7p/4b

Fig. 38 — The interbraided Turk's Head Knots.

When 'Step 1' is completed we have braided a Turk's Head Knot with 4-bights and 5-parts. This knot, the foundation Turk's Head Knot, can again be regarded as a Standard Herringbone Pineapple Knot with $A = 1$ and $B^* = 4$. Since $A = 1$ the left bight-boundary as well as the right bight-boundary of this knot carries the number 1; hence in the table which will supply us with its braiding algorithm, the value of 'L' is 1 and the value of 'R' is 1. Furthermore the value of 'A' is 1; the value of 'B^*' is 4 and the value of 's' is 5.

The position of a half-cycle from lower-left to upper-right belonging to this knot is schematically given, together with the values for L, R and A, in the left diagram of Fig. 39.

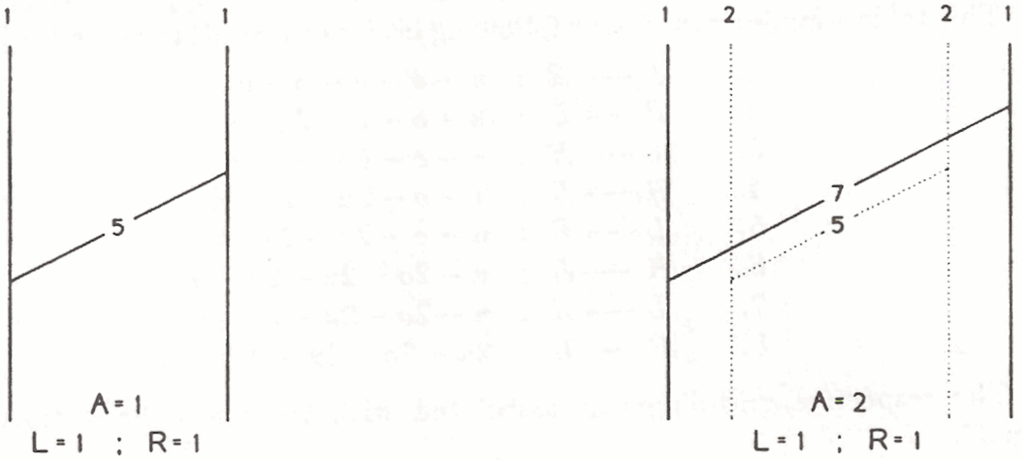


Fig. 39 — The position of a left to right half-cycle of the foundation and the interwoven Turk's Head Knots.

After substitution of $A = 1$, $L = 1$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 5$ we obtain $L - 1 = 0$, $R - 1 = 0$, $A - 1 = 0$ and hence the upper table of Fig. 40 results. This table supplies us with the following braiding algorithm associated with 'Step 1':

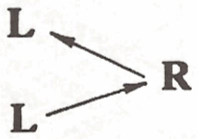
1. $L \longrightarrow R$: free run.
2. $R \longrightarrow L$: o.
3. $L \longrightarrow R$: o.
4. $R \longrightarrow L$: $u - o$.
5. $L \longrightarrow R$: $u - o$.
6. $R \longrightarrow L$: $o - u - o$.
7. $L \longrightarrow R$: $o - u - o$.
8. $R \longrightarrow L$: $u - o - u - o$.

The respective grid-diagrams associated with the above half-cycles are given in Fig. 41.

After completion of 'Step 1', we are ready to start with 'Step 2', which is the interbraiding of the Turk's Head Knot with $s = 7$ parts. When this interbraiding is completed, we have two left bight-boundaries and two right bight-boundaries; hence $A = 2$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1 and 2, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 2, and a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 1. Hence for the 'Step 2' Turk's Head Knot the value for 'L' is 1 and the value for 'R' is 1, with $A = 2$ and $s = 7$.

The positions of the half-cycles from lower-left to upper-right belonging to the 'Step 1' and 'Step 2' Turk's Head Knots are schematically illustrated, together with the values for L, R and A, in the right diagram of Fig. 39.

$B^*=4$; *Enlarging cycle*



s=5

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	0	0	0	0	0
ref. R → L	0				0
1. L → R	0	0	0	0	0
2. R → L	0	0	0	1	0
3. L → R	0	0	0	1	0
4. R → L	0	0	1	1	0
5. L → R	0	0	1	1	0
6. R → L	0	1	1	1	0
7. L → R	0	1	1	1	0
8. R → L	1	1	1	1	0

s=7

set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	0	1	1	1	1	1	0
ref. R → L	0						0
1. L → R	0	1	1	1	1	1	0
2. R → L	0	1	1	2	1	1	0
3. L → R	0	1	1	2	1	1	0
4. R → L	1	1	1	2	2	1	0
5. L → R	1	1	1	2	2	1	0
6. R → L	1	2	1	2	2	2	0
7. L → R	1	2	1	2	2	2	0
8. R → L	1	2	2	2	2	2	0

Fig. 40 — The resulting Algorithm-tables.

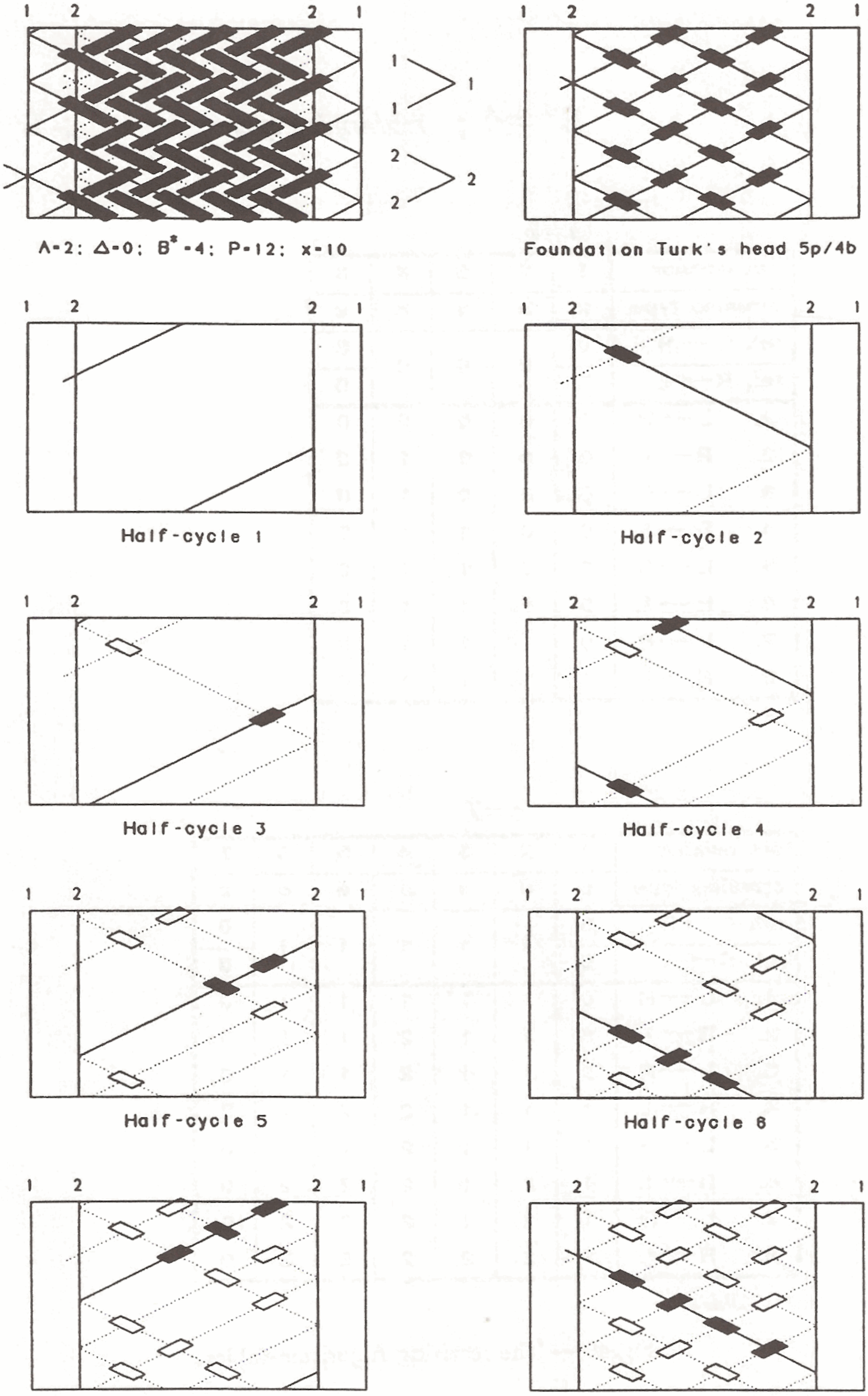
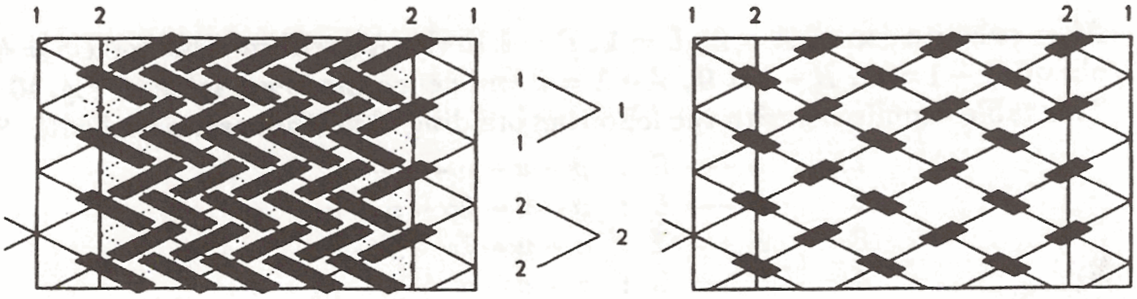
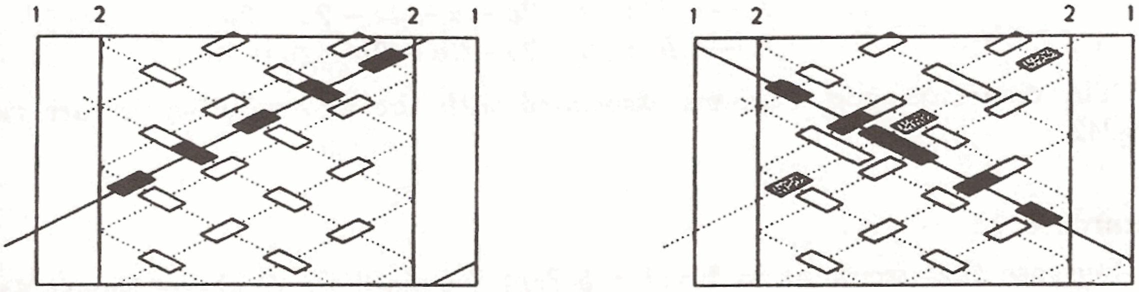


Fig. 41 — The half-cycles in the braiding of the foundation Turk's Head Knot.



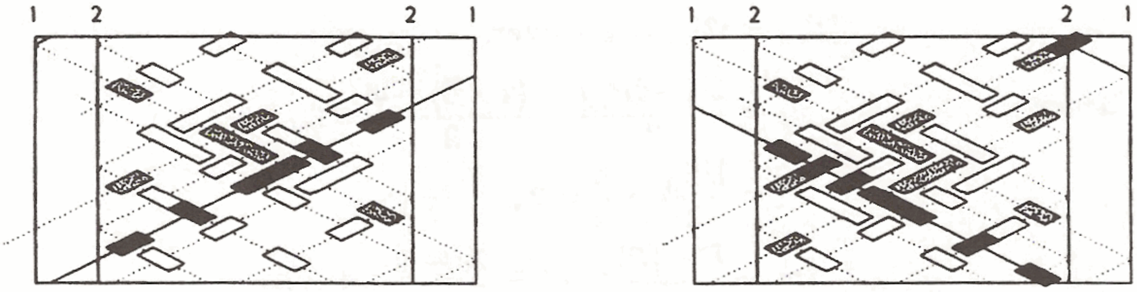
A=2; Δ=0; B* =4; P=12; x=10

Interwoven Turk's head 7p/4b



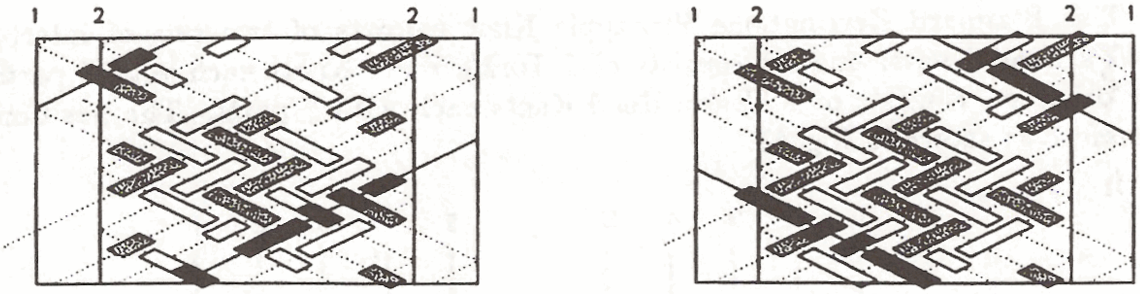
Half-cycle 1

Half-cycle 2



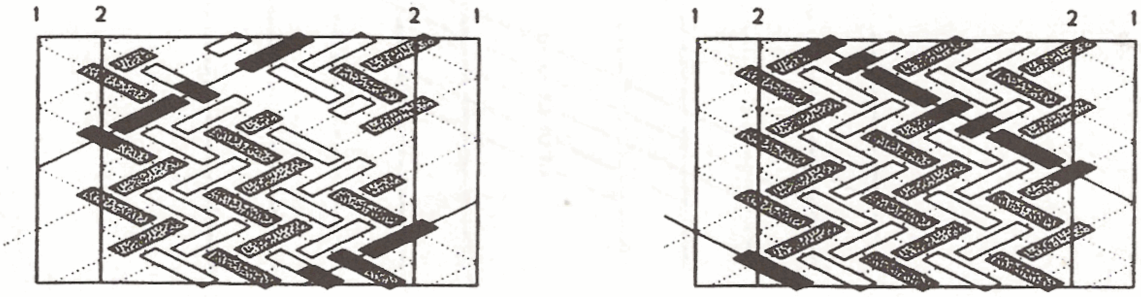
Half-cycle 3

Half-cycle 4



Half-cycle 5

Half-cycle 6



Half-cycle 7

Half-cycle 8

Fig. 42 — The half-cycles in the braiding of the interwoven Turk's Head Knot.

After substitution of $A = 2$, $L = 1$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 7$ we obtain $L - 1 = 0$, $R - 1 = 0$, $A - 1 = 1$ and hence the lower table of Fig. 40 results.

This table supplies us with the following braiding algorithm associated with 'Step 2':

1. $L \longrightarrow R \; : \; o - u - o - u - o \, .$
2. $R \longrightarrow L \; : \; o - u - 2o - u - o \, .$
3. $L \longrightarrow R \; : \; o - u - 2o - u - o \, .$
4. $R \longrightarrow L \; : \; u - o - u - 2o - 2u - o \, .$
5. $L \longrightarrow R \; : \; u - o - u - 2o - 2u - o \, .$
6. $R \longrightarrow L \; : \; u - 2o - u - 2o - 2u - 2o \, .$
7. $L \longrightarrow R \; : \; u - 2o - u - 2o - 2u - 2o \, .$
8. $R \longrightarrow L \; : \; u - 2o - 2u - 2o - 2u - 2o \, .$

The respective grid-diagrams associated with the above half-cycles are given in Fig. 42.

Example 2 :

Suppose that we want to braid a 5-Pass Standard Herringbone Pineapple Knot (hence $A = 5$), with $B^* = 4$ and $P = 29$.

Thus
$$P = 29 = \textbf{odd}\,,$$
$$|P|_A = |29|_5 = 4 = \textbf{even}\,.$$

Hence
$$a_1 = \frac{2A - |P|_A}{2} = \frac{(2 \times 5) - 4}{2} = 3\,,$$
$$a_2 = \frac{|P|_A}{2} = \frac{4}{2} = 2\,,$$
$$2m = \frac{P - |P|_A}{A} + 1 = \frac{29 - 4}{5} + 1 = 6\,.$$

Hence $\quad 2m - 1 = 5 \quad$ and $\quad 2m + 1 = 7\,.$

This Standard Herringbone Pineapple Knot consists of two sets of interbraided Turk's Head Knots; one set consists of 3 Turk's Head Knots each with 5 parts, and the other set consists of 2 Turk's Head Knots each with 7 parts. The positioning is therefore as shown in Fig. 43.

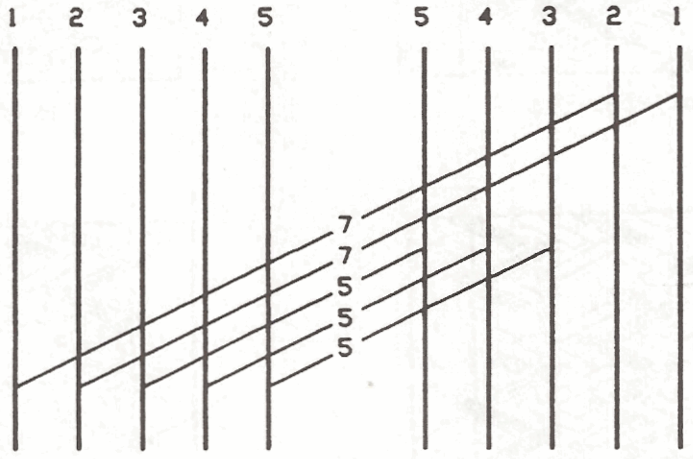


Fig. 43 — The positioning of the interbraided Turk's Head Knots.

Since $A = 5$ we can braid this Standard Herringbone Pineapple Knot in $5! = 120$ different ways. We shall discuss in detail two out of these 120 different ways. This should provide the reader with a thorough grounding in these new methods, so that he will be able to braid any Standard Herringbone Pineapple Knot with ease.

The grid-diagram (string-run and coding) of this Standard Herringbone Pineapple Knot together with the first-return string-runs of the five interbraided Turk's Head Knots (components) is given in Fig. 44. As mentioned before, it is essential that, for a good understanding, the reader draws up some grid-diagrams of different Standard Herringbone Pineapple Knots on Isometric graph paper (see Ref. [1]).

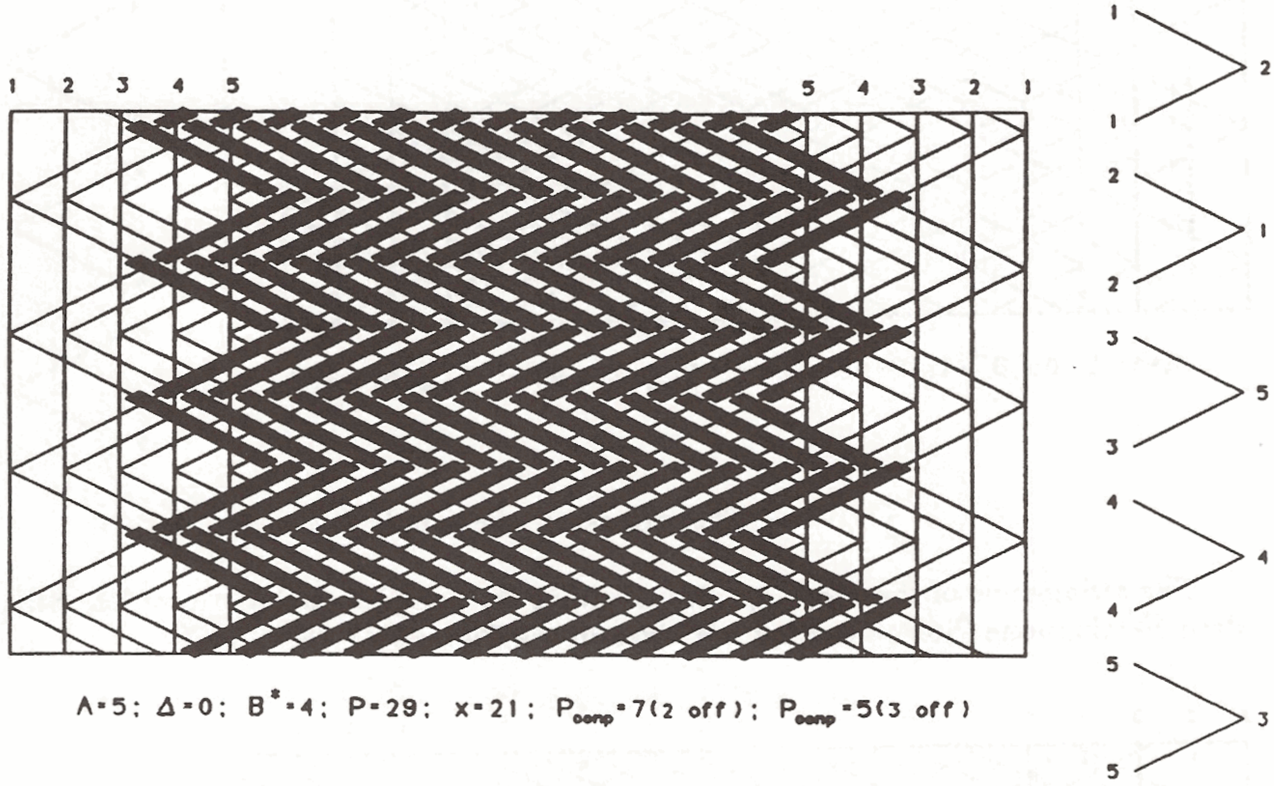

1 2 3 4 5
5 4 3 2 1
$A=5$; $\Delta=0$; $B^*=4$; $P=29$; $x=21$; $P_{comp}=7$(2 off); $P_{comp}=5$(3 off)
1 2 1
2 1 2
3 5 3
4 4 4
5 3 5

Fig. 44 — The Standard Herringbone Pineapple Knot of example 2.

Adjacent bight-boundaries are always two columns apart and the value of x can readily be obtained from the relationship $P = x + 2A - 2$, which gives $x = P - 2A + 2$. Hence in our example $x = 29 - (2 \times 5) + 2 = 21$; this means that the distance between the left bight-boundary '5' and the right bight-boundary '5' is equal to 21 columns. The distance between the top horizontal line and the bottom horizontal line is equal to $A \cdot B^*$ bight-units (1 bight-unit is equal to 2 rows); hence in our example there are $5 \times 4 = 20$ bight-units. The string-run of our example Standard Herringbone Pineapple Knot is given in Fig. 45.

(i) Let us assume that we want to braid the given Standard Herringbone Pineapple Knot in the following way:

Step 1. — Braid the component with half-cycle $1 \longrightarrow 2$.
Step 2. — Braid the component with half-cycle $2 \longrightarrow 1$.
Step 3. — Braid the component with half-cycle $3 \longrightarrow 5$.
Step 4. — Braid the component with half-cycle $4 \longrightarrow 4$.
Step 5. — Braid the component with half-cycle $5 \longrightarrow 3$.

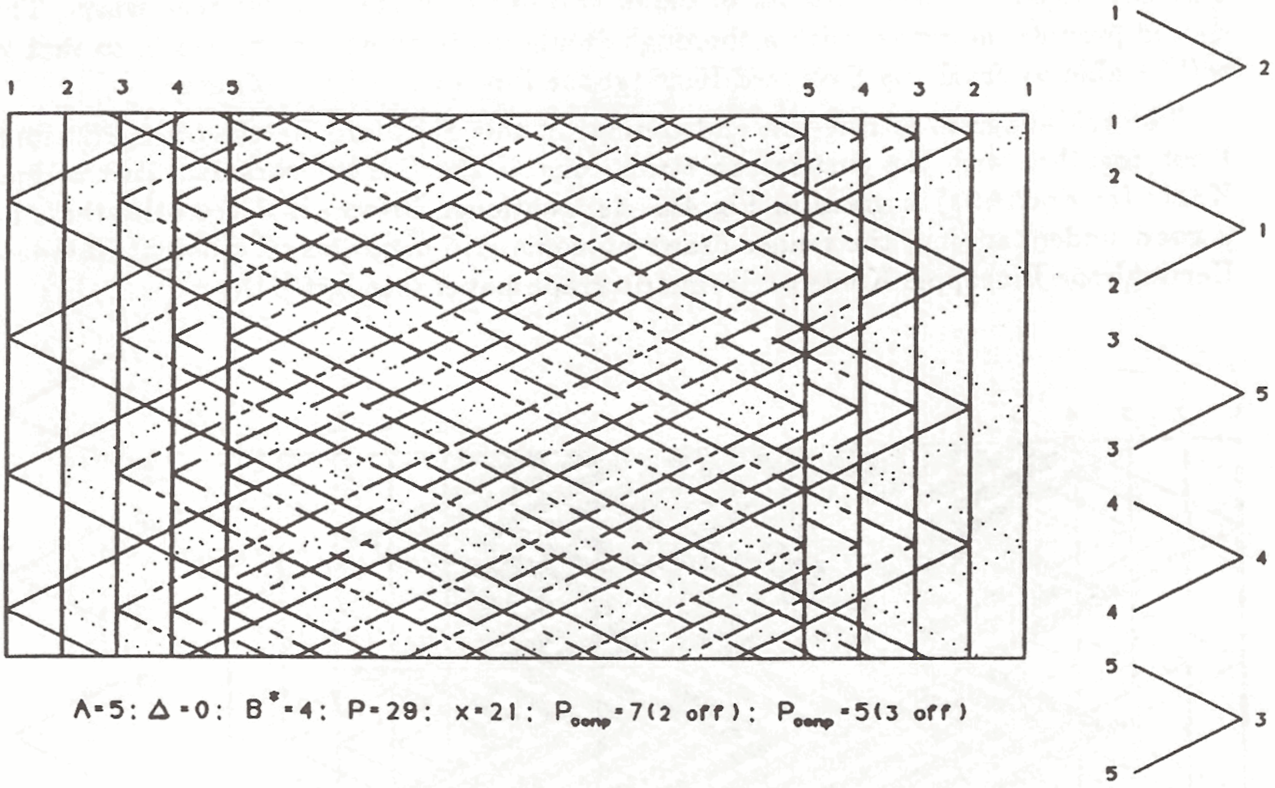

1 2 3 4 5
5 4 3 2 1
A=5: Δ=0: B*=4: P=29: x=21: P_comp=7(2 off): P_comp=5(3 off)
1
2
1
2
1
2
3
5
3
4
4
4
5
3
5

Fig. 45 — The string-run of example 2.

The string-runs of the Steps 1, 2, 3, 4, 5 within the complete string-run of this Standard Herringbone Pineapple Knot are given in the Figs. 46, 47, 48, 49, 50.

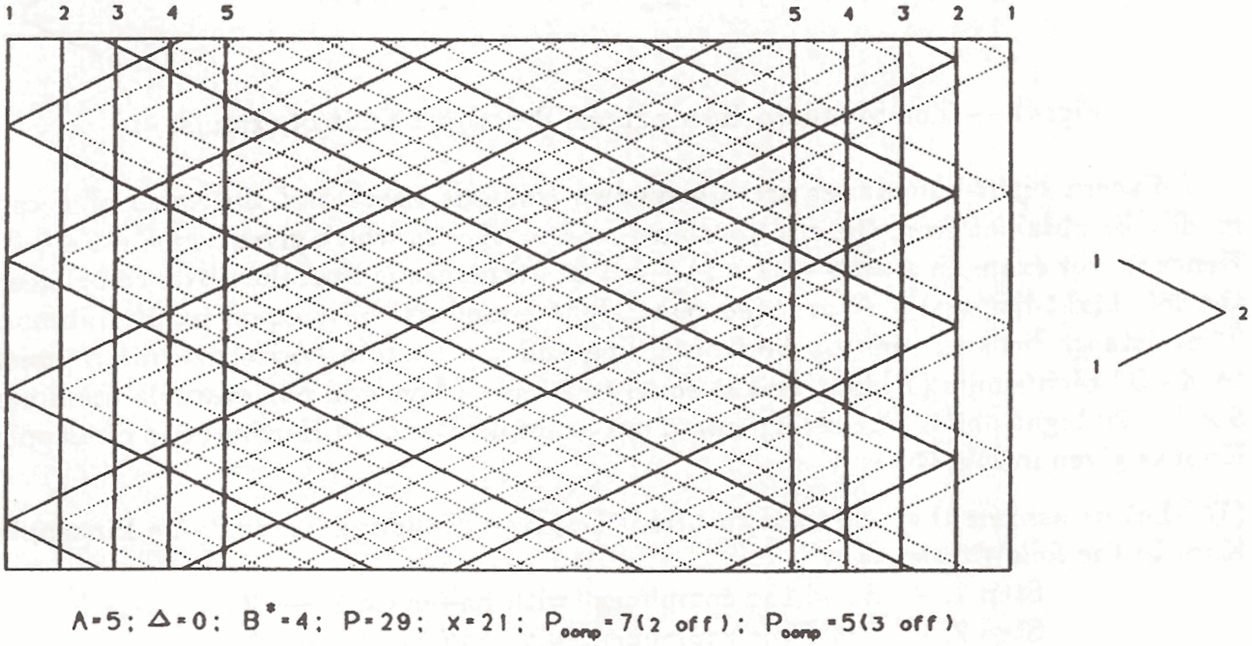
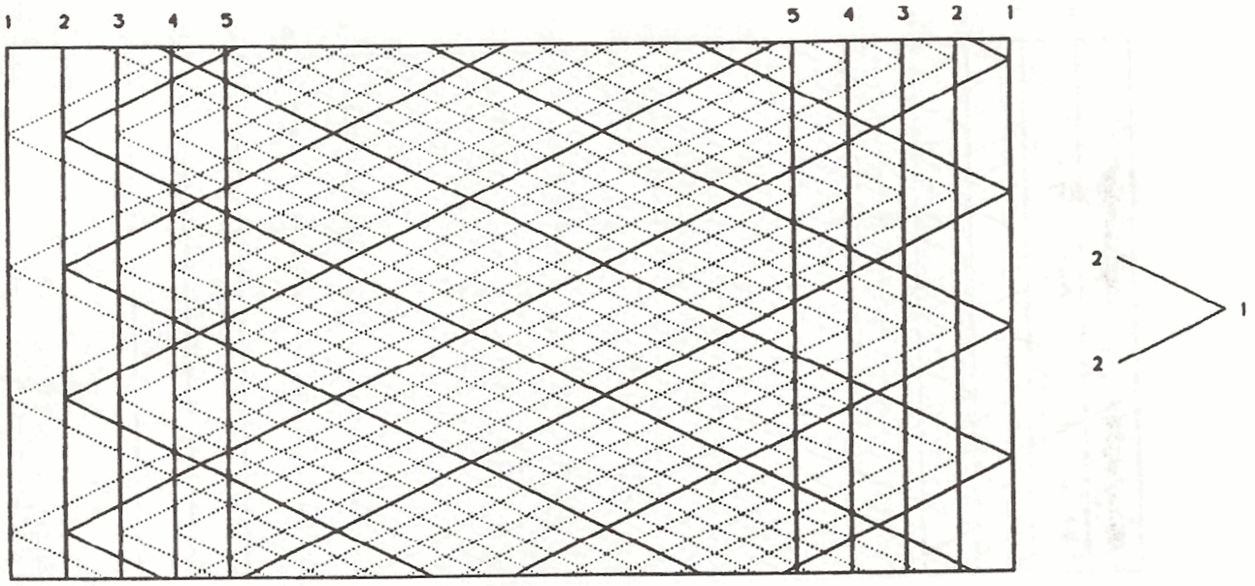
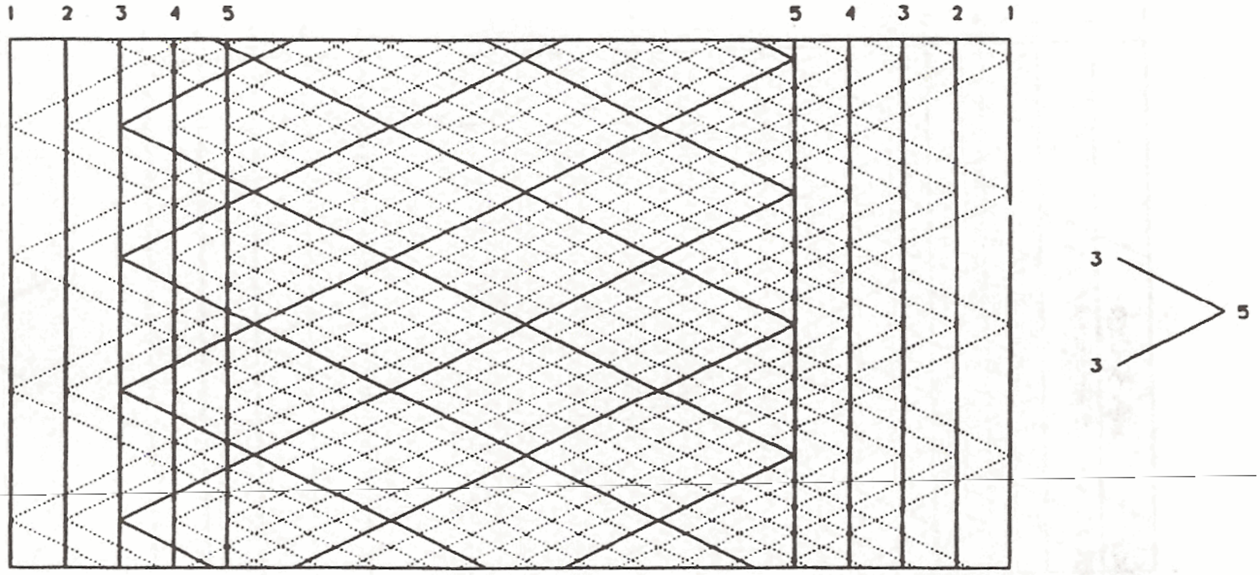

1 2 3 4 5
5 4 3 2 1
1
2
1
A=5: Δ=0: B*=4: P=29: x=21: P_comp=7(2 off): P_comp=5(3 off)

Fig. 46 — String-run of Step 1 (solid line).



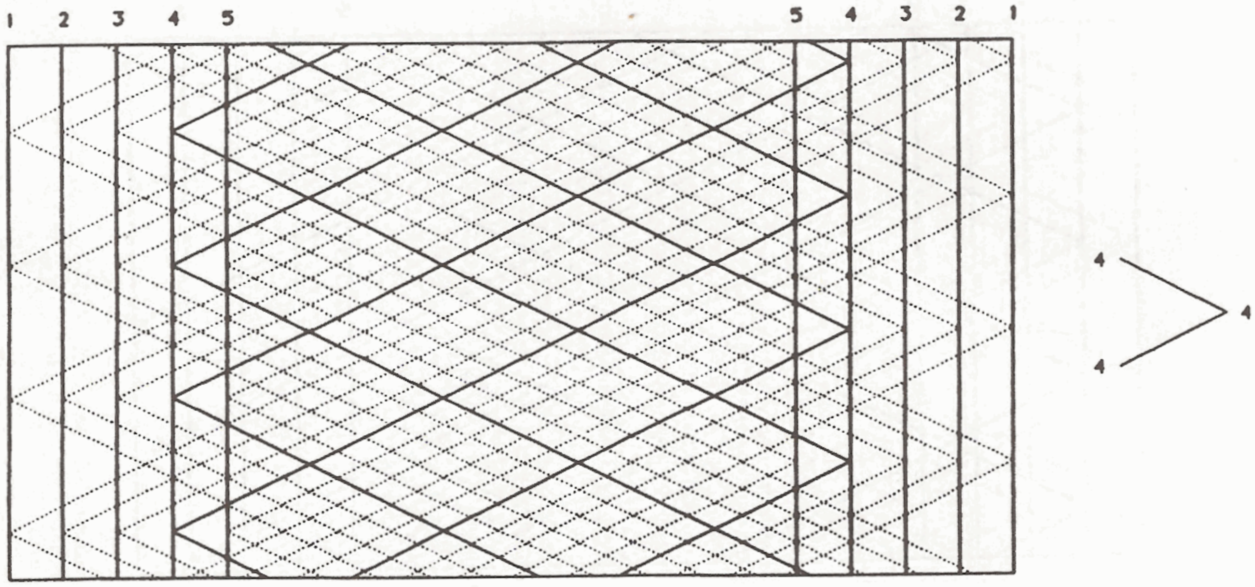
$A=5$; $\Delta=0$; $B^*=4$; $P=29$; $x=21$; $P_{comp}=7$(2 off); $P_{comp}=5$(3 off)

Fig. 47 — String-run of Step 2 (solid line).



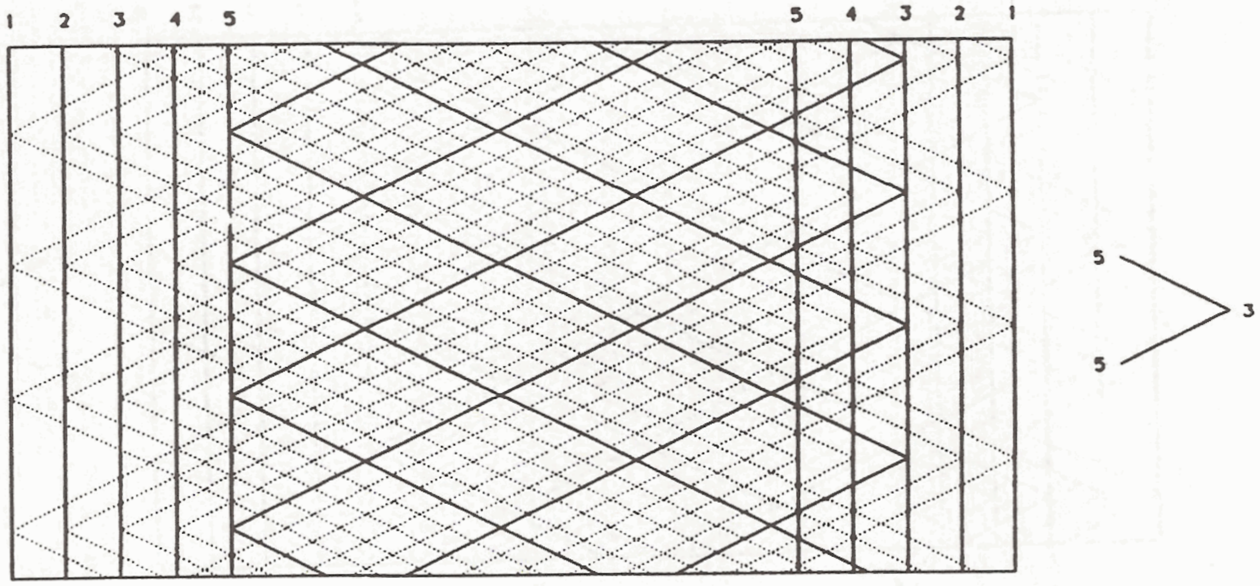
$A=5$; $\Delta=0$; $B^*=4$; $P=29$; $x=21$; $P_{comp}=7$(2 off); $P_{comp}=5$(3 off)

Fig. 48 — String-run of Step 3 (solid line).



$A=5;\ \Delta=0;\ B^*=4;\ P=29;\ x=21;\ P_{comp}=7(2\ off);\ P_{comp}=5(3\ off)$

Fig. 49 — String-run of Step 4 (solid line).



$A=5;\ \Delta=0;\ B^*=4;\ P=29;\ x=21;\ P_{comp}=7(2\ off);\ P_{comp}=5(3\ off)$

Fig. 50 — String-run of Step 5 (solid line).

When 'Step 1' is completed we have braided a Turk's Head Knot with 4-bights and 7-parts. This knot can also be regarded as a Standard Herringbone Pineapple Knot with $A = 1$ and $B^* = 4$. Since $A = 1$ the left bight-boundary as well as the right bight-boundary of this knot carries the number 1; hence in the table which will supply us with its braiding algorithm, the value of 'L' is 1 and the value of 'R' is 1. Furthermore the value of 'A' is 1; the value of 'B^*' is 4 and the value of 's' is 7.

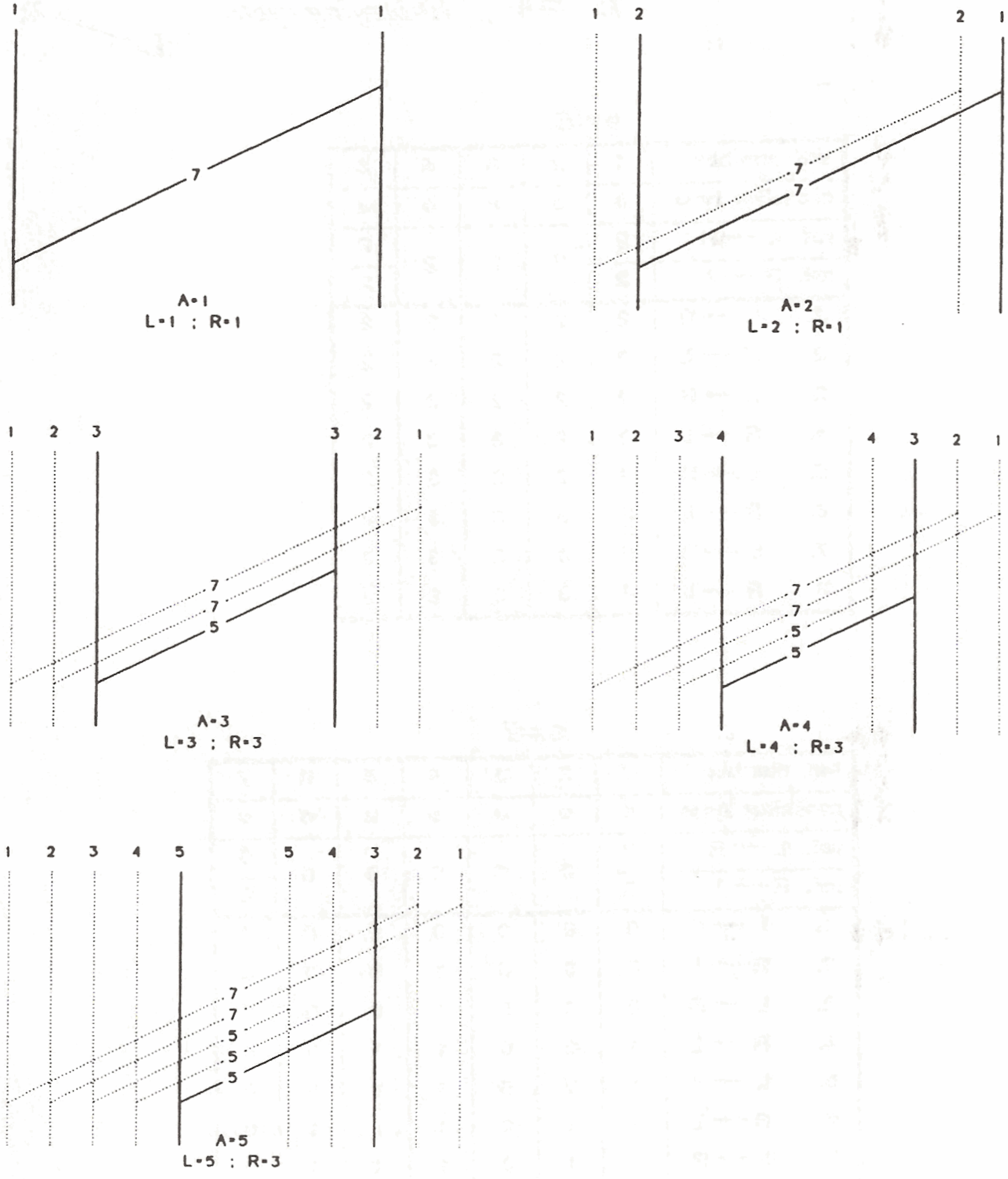
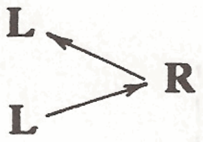


Fig. 51 — The positioning of the left to right half-cycles in the five Steps.

The position of a half-cycle from lower-left to upper-right belonging to this knot is schematically given, together with the values for L, R and A, in the upper left diagram of Fig. 51.

After substitution of $A = 1$, $L = 1$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 7$ we obtain $L - 1 = 0$, $R - 1 = 0$, $A - 1 = 0$ and hence the lower table of Fig. 52 results.

$B^*=4$; *Enlarging cycle*



s=5

set number		1	2	3	4	3
crossing type		u	o	u	o	u
ref. L → R		2	2	2	2	2
ref. R → L		2				2
1.	L → R	2	2	2	2	2
2.	R → L	2	2	2	3	2
3.	L → R	2	2	2	3	2
4.	R → L	2	2	3	3	2
5.	L → R	2	2	3	3	2
6.	R → L	2	3	3	3	2
7.	L → R	2	3	3	3	2
8.	R → L	3	3	3	3	2

s=7

set number		1	2	3	4	5	6	7
crossing type		u	o	u	o	u	o	u
ref. L → R		0	0	0	0	0	0	0
ref. R → L		0						0
1.	L → R	0	0	0	0	0	0	0
2.	R → L	0	0	0	1	0	0	0
3.	L → R	0	0	0	1	0	0	0
4.	R → L	1	0	0	1	1	0	0
5.	L → R	1	0	0	1	1	0	0
6.	R → L	1	1	0	1	1	1	0
7.	L → R	1	1	0	1	1	1	0
8.	R → L	1	1	1	1	1	1	0

Fig. 52 — Algorithm-tables for Step 1 and Step 3.

This table supplies us with the following braiding algorithm associated with 'Step 1':

1. $L \longrightarrow R$: free run.
2. $R \longrightarrow L$: o .
3. $L \longrightarrow R$: o .
4. $R \longrightarrow L$: $u - o - u$.
5. $L \longrightarrow R$: $u - o - u$.
6. $R \longrightarrow L$: $u - 2o - u - o$.
7. $L \longrightarrow R$: $u - 2o - u - o$.
8. $R \longrightarrow L$: $u - o - u - o - u - o$.

After completion of 'Step 1', we are ready to start with 'Step 2', which is the interbraiding of the Turk's Head Knot with $s = 7$ parts. When this interbraiding is completed, we have two left bight-boundaries and two right bight-boundaries; hence $A = 2$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1 and 2, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot' runs from the left bight-boundary 1 to the right bight-boundary 2, and a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot' runs from the left bight-boundary 2 to the right bight-boundary 1. Hence for the 'Step 2' Turk's Head Knot' the value for 'L' is 2 and the value for 'R' is 1, with $A = 2$ and $s = 7$.

The positions of the half-cycles from lower-left to upper-right belonging to the 'Step 1' and 'Step 2' Turk's Head Knots are schematically illustrated, together with the values for L , R and A , in the upper-right diagram of Fig. 51.

After substitution of $A = 2$, $L = 2$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 7$ we obtain $L - 1 = 1$, $R - 1 = 0$, $A - 1 = 1$ and hence the lower table of Fig. 53 results.

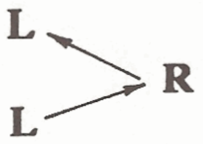
This table supplies us with the following braiding algorithm associated with 'Step 2':

1. $L \longrightarrow R$: $u - o - u - o - u - o$.
2. $R \longrightarrow L$: $o - u - 2o - u - o - u$.
3. $L \longrightarrow R$: $u - o - u - 2o - u - o$.
4. $R \longrightarrow L$: $u - o - u - 2o - 2u - o - u$.
5. $L \longrightarrow R$: $2u - o - u - 2o - 2u - o$.
6. $R \longrightarrow L$: $u - 2o - u - 2o - 2u - 2o - u$.
7. $L \longrightarrow R$: $2u - 2o - u - 2o - 2u - 2o$.
8. $R \longrightarrow L$: $u - 2o - 2u - 2o - 2u - 2o - u$.

After completion of 'Step 2', we are ready to start with 'Step 3', which is the interbraiding of the Turk's Head Knot with $s = 5$ parts. When this interbraiding is completed, we have three left bight-boundaries and three right bight-boundaries; hence $A = 3$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1, 2 and 3, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 2, a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 1, and a lower-left to upper-right half-cycle of the 'Step 3' Turk's Head Knot runs from the left bight-boundary 3 to the right bight-boundary 3. Hence for the 'Step 3' Turk's Head Knot the value for 'L' is 3 and the value for 'R' is 3, with $A = 3$ and $s = 5$.

The positions of the half-cycles from lower-left to upper-right belonging to the Steps 1, 2 and 3 Turk's Head Knots are schematically illustrated, together with the values for L , R and A , in the left diagram of the centre row in Fig. 51.

$B^*=4$; *Enlarging cycle*



s=5

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	3	3	3	3	2
ref. R → L	2				3
1. L → R	3	3	3	3	2
2. R → L	2	3	3	4	3
3. L → R	3	3	3	4	2
4. R → L	2	3	4	4	3
5. L → R	3	3	4	4	2
6. R → L	2	4	4	4	3
7. L → R	3	4	4	4	2
8. R → L	3	4	4	4	3

s=7

set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	1	1	1	1	1	1	0
ref. R → L	0						1
1. L → R	1	1	1	1	1	1	0
2. R → L	0	1	1	2	1	1	1
3. L → R	1	1	1	2	1	1	0
4. R → L	1	1	1	2	2	1	1
5. L → R	2	1	1	2	2	1	0
6. R → L	1	2	1	2	2	2	1
7. L → R	2	2	1	2	2	2	0
8. R → L	1	2	2	2	2	2	1

Fig. 53 — Algorithm-tables for Step 2 and Step 4.

After substitution of $A = 3$, $L = 3$, $R = 3$ in the Algorithm-table for $B^* = 4$, $s = 5$ we obtain $L - 1 = 2$, $R - 1 = 2$, $A - 1 = 2$ and hence the upper table of Fig. 52 results.

This table supplies us with the following braiding algorithm associated with 'Step 3':

1. $L \longrightarrow R$: $2u - 2o - 2u - 2o - 2u$.
2. $R \longrightarrow L$: $2u - 2o - 2u - 3o - 2u$.
3. $L \longrightarrow R$: $2u - 2o - 2u - 3o - 2u$.
4. $R \longrightarrow L$: $2u - 2o - 3u - 3o - 2u$.
5. $L \longrightarrow R$: $2u - 2o - 3u - 3o - 2u$.
6. $R \longrightarrow L$: $2u - 3o - 3u - 3o - 2u$.
7. $L \longrightarrow R$: $2u - 3o - 3u - 3o - 2u$.
8. $R \longrightarrow L$: $3u - 3o - 3u - 3o - 2u$.

After completion of 'Step 3', we are ready to start with 'Step 4', which is the interbraiding of the Turk's Head Knot with $s = 5$ parts. When this interbraiding is completed, we have four left bight-boundaries and four right bight-boundaries; hence $A = 4$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1, 2, 3 and 4, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 2, a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 1, a lower-left to upper-right half-cycle of the 'Step 3' Turk's Head Knot runs from the left bight-boundary 3 to the right bight-boundary 4, and a lower-left to upper-right half-cycle of the 'Step 4' Turk's Head Knot runs from the left bight-boundary 4 to the right bight-boundary 3. Hence for the 'Step 4' Turk's Head Knot the value for 'L' is 4 and the value for 'R' is 3, with $A = 4$ and $s = 5$.

The positions of the half-cycles from lower-left to upper-right belonging to the Steps 1, 2, 3 and 4 Turk's Head Knots are schematically illustrated, together with the values for L, R and A, in the right diagram of the centre row in Fig. 51.

After substitution of $A = 4$, $L = 4$, $R = 3$ in the Algorithm-table for $B^* = 4$, $s = 5$ we obtain $L - 1 = 3$, $R - 1 = 2$, $A - 1 = 3$ and hence the upper table of Fig. 53 results.

This table supplies us with the following braiding algorithm associated with 'Step 4':

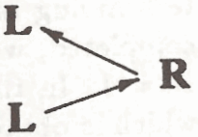
1. $L \longrightarrow R$: $3u - 3o - 3u - 3o - 2u$.
2. $R \longrightarrow L$: $2u - 3o - 3u - 4o - 3u$.
3. $L \longrightarrow R$: $3u - 3o - 3u - 4o - 2u$.
4. $R \longrightarrow L$: $2u - 3o - 4u - 4o - 3u$.
5. $L \longrightarrow R$: $3u - 3o - 4u - 4o - 2u$.
6. $R \longrightarrow L$: $2u - 4o - 4u - 4o - 3u$.
7. $L \longrightarrow R$: $3u - 4o - 4u - 4o - 2u$.
8. $R \longrightarrow L$: $3u - 4o - 4u - 4o - 3u$.

After completion of 'Step 4', we are ready to start with 'Step 5', which is the interbraiding of the Turk's Head Knot with $s = 5$ parts. When this interbraiding is completed, we have five left bight-boundaries and five right bight-boundaries; hence $A = 5$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1, 2, 3, 4 and 5, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 2, a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 1, a lower-left to upper-right half-cycle of the 'Step 3' Turk's Head Knot runs from the left bight-boundary 3 to the right bight-boundary 5, a lower-left to upper-right half-cycle of the 'Step 4' Turk's Head Knot runs from the left bight-boundary 4 to the right bight-boundary 4, and a lower-left to upper-right half-cycle of the 'Step 5' Turk's Head Knot runs from the left bight-boundary 5 to the right bight-boundary 3. Hence for the 'Step 5' Turk's Head Knot the value for 'L' is 5 and the value for 'R' is 3, with $A = 5$ and $s = 5$.

The positions of the half-cycles from lower-left to upper-right belonging to the Steps 1, 2, 3, 4 and 5 Turk's Head Knots are schematically illustrated, together with the values for L, R and A, in the lower left diagram of Fig. 51.

After substitution of $A = 5$, $L = 5$, $R = 3$ in the Algorithm-table for $B^* = 4$, $s = 5$ we obtain $L - 1 = 4$, $R - 1 = 2$, $A - 1 = 4$ and hence the table of Fig. 54 results.

B*=4 ; *Enlarging cycle*



s=5

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	4	4	4	4	2
ref. R → L	2				4
1. L → R	4	4	4	4	2
2. R → L	2	4	4	5	4
3. L → R	4	4	4	5	2
4. R → L	2	4	5	5	4
5. L → R	4	4	5	5	2
6. R → L	2	5	5	5	4
7. L → R	4	5	5	5	2
8. R → L	3	5	5	5	4

Fig. 54 — Algorithm-tables for Step 5.

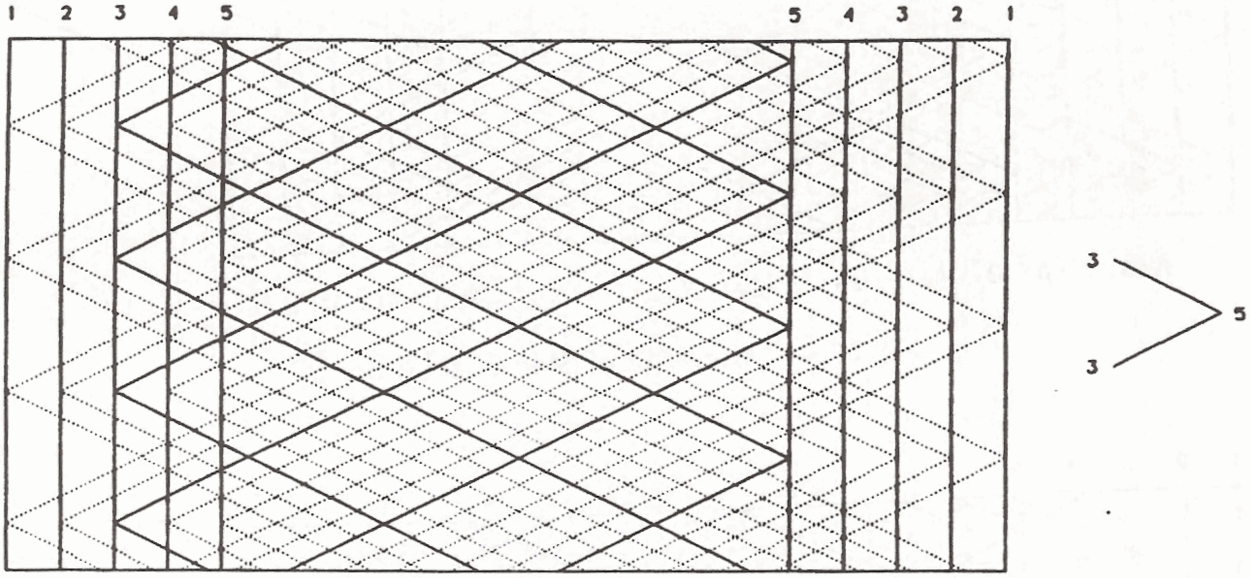
This table supplies us with the following braiding algorithm associated with 'Step 5':

1. $L \longrightarrow R$: $4u - 4o - 4u - 4o - 2u$.
2. $R \longrightarrow L$: $2u - 4o - 4u - 5o - 4u$.
3. $L \longrightarrow R$: $4u - 4o - 4u - 5o - 2u$.
4. $R \longrightarrow L$: $2u - 4o - 5u - 5o - 4u$.
5. $L \longrightarrow R$: $4u - 4o - 5u - 5o - 2u$.
6. $R \longrightarrow L$: $2u - 5o - 5u - 5o - 4u$.
7. $L \longrightarrow R$: $4u - 5o - 5u - 5o - 2u$.
8. $R \longrightarrow L$: $3u - 5o - 5u - 5o - 4u$.

(ii) Let us now assume that we want to braid the given Standard Herringbone Pineapple Knot in the following way:

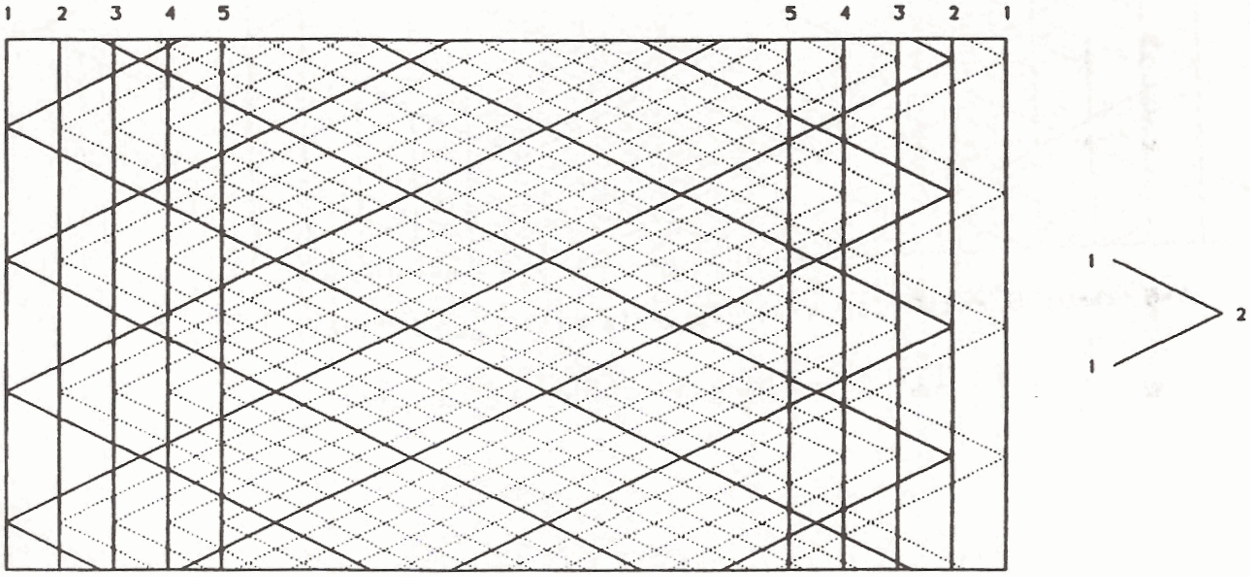
Step 1. — Braid the component with half-cycle 3 ⟶ 5 .
Step 2. — Braid the component with half-cycle 1 ⟶ 2 .
Step 3. — Braid the component with half-cycle 5 ⟶ 3 .
Step 4. — Braid the component with half-cycle 2 ⟶ 1 .
Step 5. — Braid the component with half-cycle 4 ⟶ 4 .

The string-runs of the Steps 1, 2, 3, 4, 5 within the complete string-run of this Standard Herringbone Pineapple Knot are given in the Figs. 55, 56, 57, 58, 59.



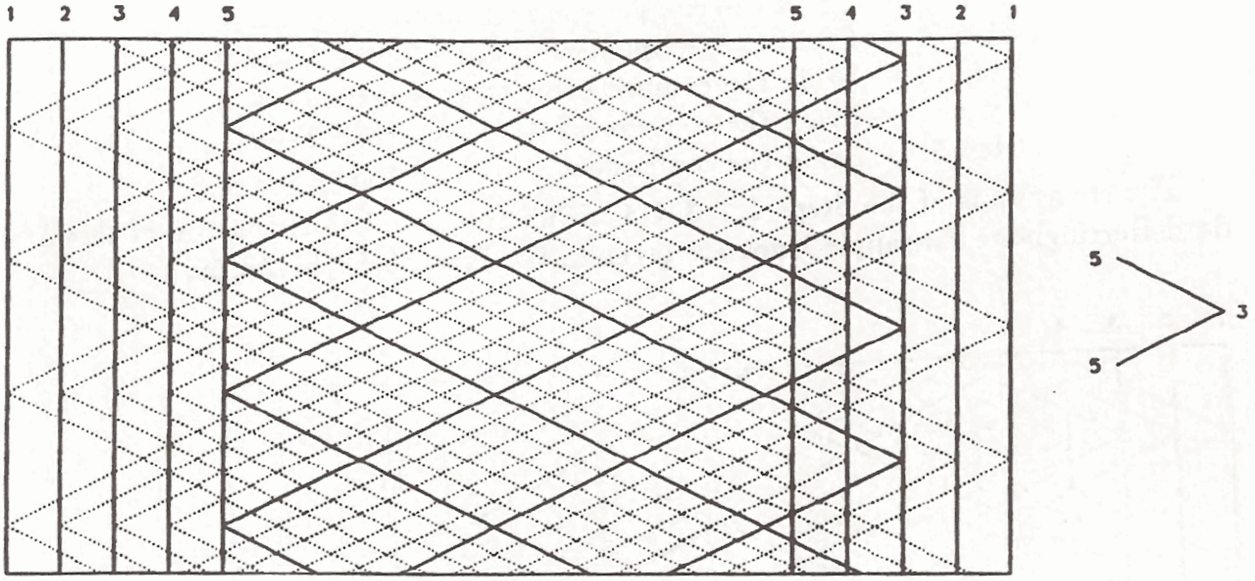
$A=5$; $\Delta=0$; $B^*=4$; $P=29$; $x=21$; $P_{comp}=7$(2 off); $P_{comp}=5$(3 off)

Fig. 55 — String-run of Step 1 (solid line).



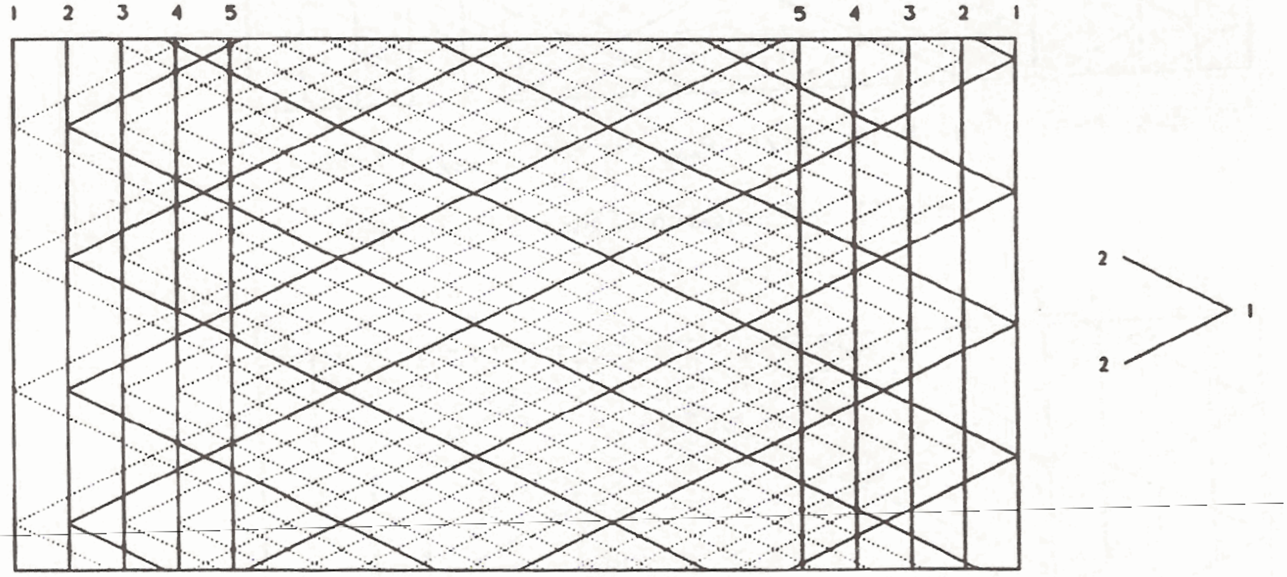
$A=5$; $\Delta=0$; $B^*=4$; $P=29$; $x=21$; $P_{comp}=7$(2 off); $P_{comp}=5$(3 off)

Fig. 56 — String-run of Step 2 (solid line).



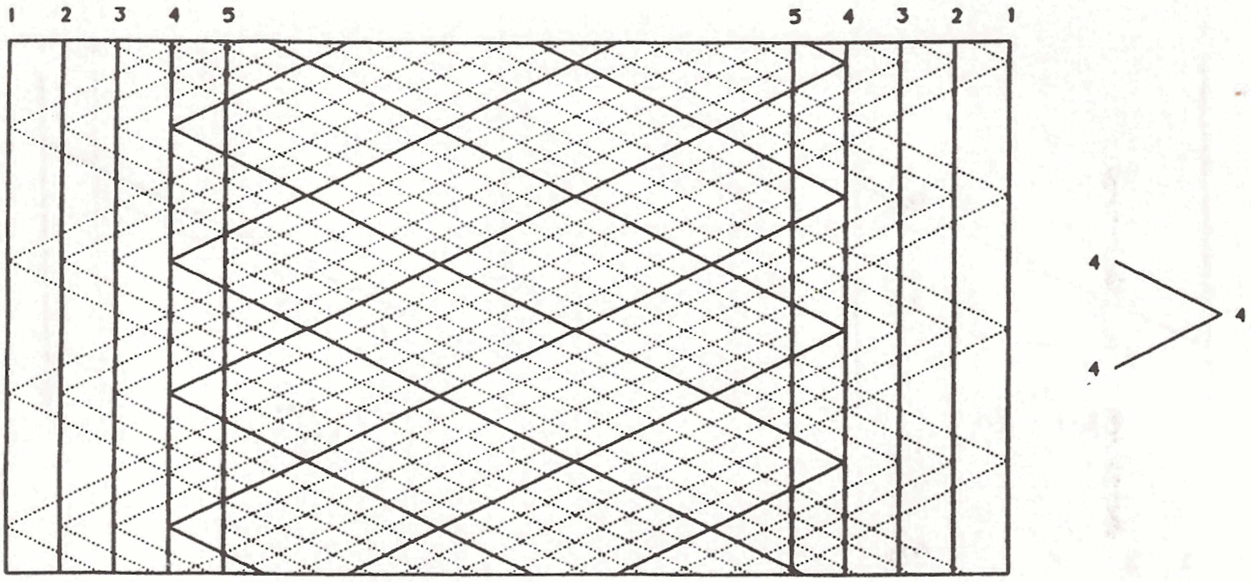
$A=5$: $\Delta=0$: $B^*=4$: $P=29$: $x=21$: $P_{comp}=7$(2 off): $P_{comp}=5$(3 off)

Fig. 57 — String-run of Step 3 (solid line).



$A=5$; $\Delta=0$; $B^*=4$; $P=29$; $x=21$: $P_{comp}=7$(2 off); $P_{comp}=5$(3 off)

Fig. 58 — String-run of Step 4 (solid line).



$A = 5;\ \Delta = 0;\ B^* = 4;\ P = 28;\ x = 21;\ P_{comp} = 7(2\ off);\ P_{comp} = 5(3\ off)$

Fig. 59 — String-run of Step 5 (solid line).

When 'Step 1' is completed we have braided a Turk's Head Knot with 4-bights and 5-parts. This knot can again be seen as a Standard Herringbone Pineapple Knot with $A = 1$ and $B^* = 4$. Since $A = 1$ the left bight-boundary as well as the right bight-boundary of this knot carries the number 1; hence in the table which will supply us with its braiding algorithm, the value of 'L' is 1 and the value of 'R' is 1. Furthermore the value of 'A' is 1; the value of 'B^*' is 4 and the value of 's' is 5.

The position of a half-cycle from lower-left to upper-right belonging to this knot is schematically given, together with the values for L, R and A, in the upper left diagram of Fig. 60.

After substitution of $A = 1$, $L = 1$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 5$ we obtain $L - 1 = 0$, $R - 1 = 0$, $A - 1 = 0$ and hence the upper table of Fig. 61 results.

This table supplies us with the following braiding algorithm associated with 'Step 1':

1. $L \longrightarrow R$: free run.
2. $R \longrightarrow L$: o.
3. $L \longrightarrow R$: o.
4. $R \longrightarrow L$: $u - o$.
5. $L \longrightarrow R$: $u - o$.
6. $R \longrightarrow L$: $o - u - o$.
7. $L \longrightarrow R$: $o - u - o$.
8. $R \longrightarrow L$: $u - o - u - o$.

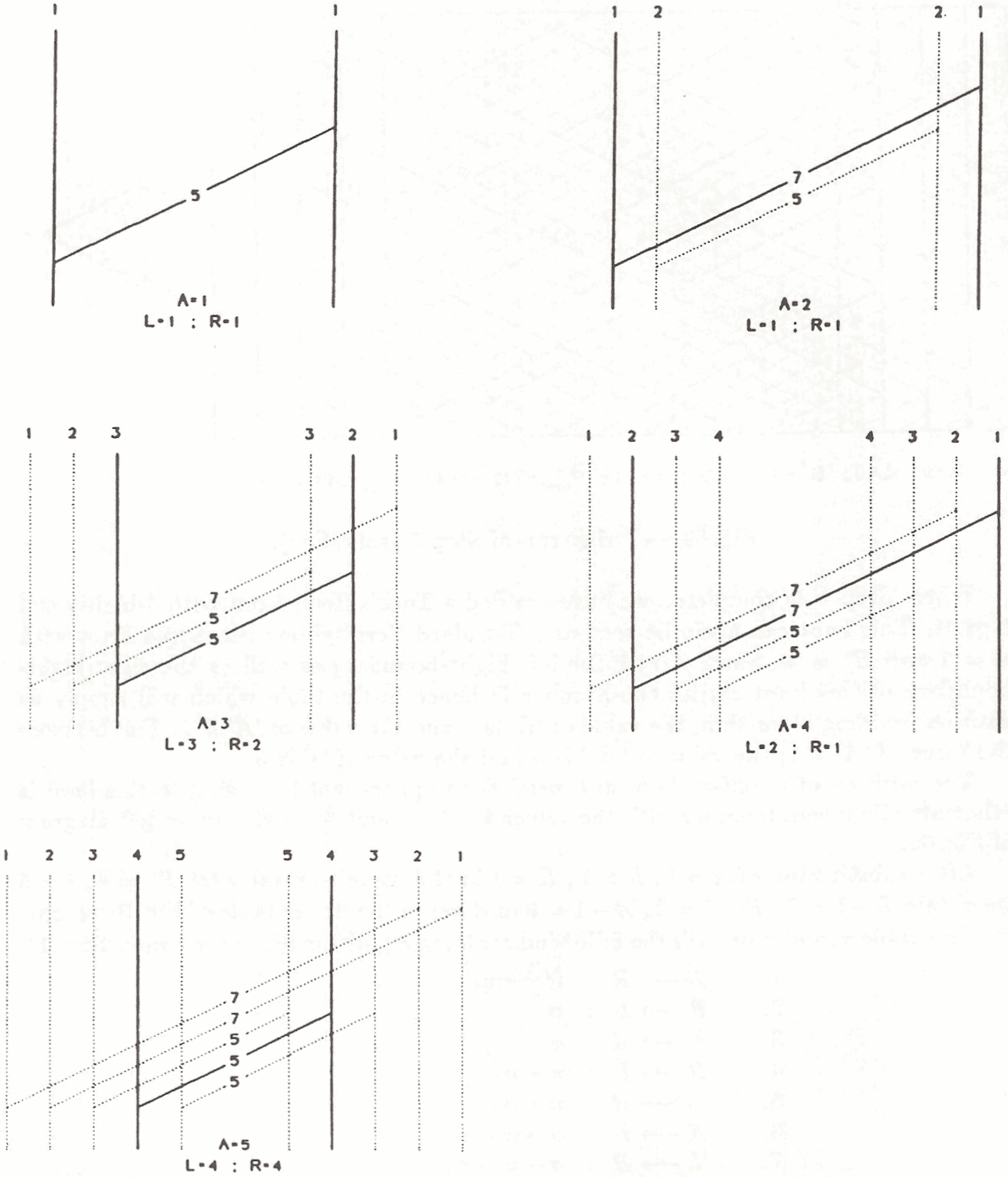
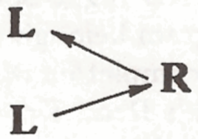


Fig. 60 — The positioning of the left to right half-cycles in the five Steps.

$B^* = 4$; *Enlarging cycle*



s=5

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	0	0	0	0	0
ref. R → L	0				0
1. L → R	0	0	0	0	0
2. R → L	0	0	0	1	0
3. L → R	0	0	0	1	0
4. R → L	0	0	1	1	0
5. L → R	0	0	1	1	0
6. R → L	0	1	1	1	0
7. L → R	0	1	1	1	0
8. R → L	1	1	1	1	0

s=7

set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	0	1	1	1	1	1	0
ref. R → L	0						0
1. L → R	0	1	1	1	1	1	0
2. R → L	0	1	1	2	1	1	0
3. L → R	0	1	1	2	1	1	0
4. R → L	1	1	1	2	2	1	0
5. L → R	1	1	1	2	2	1	0
6. R → L	1	2	1	2	2	2	0
7. L → R	1	2	1	2	2	2	0
8. R → L	1	2	2	2	2	2	0

Fig. 61 — Algorithm-tables for Step 1 and Step 2.

After completion of 'Step 1', we are ready to start with 'Step 2': the interbraiding of the Turk's Head Knot with $s = 7$ parts. When this interbraiding is completed, we have two left bight-boundaries and two right bight-boundaries; hence $A = 2$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1 and 2, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 2, and a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 1. Hence for the 'Step 2' Turk's Head Knot the value for 'L' is 1 and the value for 'R' is 1, with $A = 2$ and $s = 7$.

The positions of the half-cycles from lower-left to upper-right belonging to the 'Step 1' and 'Step 2' Turk's Head Knots are schematically illustrated, together with the values for L, R and A, in the upper-right diagram of Fig. 60.

After substitution of $A = 2$, $L = 1$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 7$ we obtain $L - 1 = 0$, $R - 1 = 0$, $A - 1 = 1$ and hence the lower table of Fig. 61 results.

This table supplies us with the following braiding algorithm associated with 'Step 2':

1. $L \longrightarrow R$: $o - u - o - u - o$.
2. $R \longrightarrow L$: $o - u - 2o - u - o$.
3. $L \longrightarrow R$: $o - u - 2o - u - o$.
4. $R \longrightarrow L$: $u - o - u - 2o - 2u - o$.
5. $L \longrightarrow R$: $u - o - u - 2o - 2u - o$.
6. $R \longrightarrow L$: $u - 2o - u - 2o - 2u - 2o$.
7. $L \longrightarrow R$: $u - 2o - u - 2o - 2u - 2o$.
8. $R \longrightarrow L$: $u - 2o - 2u - 2o - 2u - 2o$.

After completion of 'Step 2', we are ready to start with 'Step 3', which is the interbraiding of the Turk's Head Knot with $s = 5$ parts. When this interbraiding is completed, we have three left bight-boundaries and three right bight-boundaries; hence $A = 3$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1,2 and 3, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 3, a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 1, and a lower-left to upper-right half-cycle of the 'Step 3' Turk's Head Knot runs from the left bight-boundary 3 to the right bight-boundary 2. Hence for the 'Step 3' Turk's Head Knot the value for 'L' is 3 and the value for 'R' is 2, with $A = 3$ and $s = 5$.

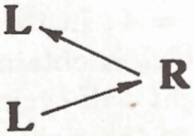
The positions of the half-cycles from lower-left to upper-right belonging to the Steps 1,2 and 3 Turk's Head Knots are schematically illustrated, together with the values for L, R and A, in the left diagram of the centre row in Fig. 60.

After substitution of $A = 3$, $L = 3$, $R = 2$ in the Algorithm-table for $B^* = 4$, $s = 5$ we obtain $L - 1 = 2$, $R - 1 = 1$, $A - 1 = 2$ and hence the upper table of Fig. 62 results.

This table supplies us with the following braiding algorithm associated with 'Step 3':

1. $L \longrightarrow R$: $2u - 2o - 2u - 2o - u$.
2. $R \longrightarrow L$: $u - 2o - 2u - 3o - 2u$.
3. $L \longrightarrow R$: $2u - 2o - 2u - 3o - u$.
4. $R \longrightarrow L$: $u - 2o - 3u - 3o - 2u$.
5. $L \longrightarrow R$: $2u - 2o - 3u - 3o - u$.
6. $R \longrightarrow L$: $u - 3o - 3u - 3o - 2u$.
7. $L \longrightarrow R$: $2u - 3o - 3u - 3o - u$.
8. $R \longrightarrow L$: $2u - 3o - 3u - 3o - 2u$.

$B^*=4$; ***Enlarging cycle***



s=5

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	2	2	2	2	1
ref. R → L	1				2
1. L → R	2	2	2	2	1
2. R → L	1	2	2	3	2
3. L → R	2	2	2	3	1
4. R → L	1	2	3	3	2
5. L → R	2	2	3	3	1
6. R → L	1	3	3	3	2
7. L → R	2	3	3	3	1
8. R → L	2	3	3	3	2

s=7

set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	1	3	3	3	3	3	0
ref. R → L	0						1
1. L → R	1	3	3	3	3	3	0
2. R → L	0	3	3	4	3	3	1
3. L → R	1	3	3	4	3	3	0
4. R → L	1	3	3	4	4	3	1
5. L → R	2	3	3	4	4	3	0
6. R → L	1	4	3	4	4	4	1
7. L → R	2	4	3	4	4	4	0
8. R → L	1	4	4	4	4	4	1

Fig. 62 — Algorithm-tables for Step 3 and Step 4.

After completion of 'Step 3', we are ready to start with 'Step 4', which is the interbraiding of the Turk's Head Knot with $s = 7$ parts. When this interbraiding is completed, we have four left bight-boundaries and four right bight-boundaries; hence $A = 4$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps $1, 2, 3$ and 4, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 3 to the right bight-boundary 4, a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 2, a lower-left to upper-right half-cycle of the 'Step 3' Turk's Head Knot runs from the left bight-boundary 4 to the right bight-boundary 3, and a lower-left to upper-right half-cycle of the 'Step 4' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 1. Hence for the 'Step 4' Turk's Head Knot the value for 'L' is 2 and the value for 'R' is 1, with $A = 4$ and $s = 7$.

The positions of the half-cycles from lower-left to upper-right belonging to the Steps $1, 2, 3$ and 4 Turk's Head Knots are schematically illustrated, together with the values for L, R and A, in the right diagram of the centre row in Fig. 60.

After substitution of $A = 4$, $L = 2$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 7$ we obtain $L - 1 = 1$, $R - 1 = 0$, $A - 1 = 3$ and hence the lower table of Fig. 62 results.

This table supplies us with the following braiding algorithm associated with 'Step 4':

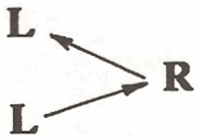
1. $L \longrightarrow R$: $u - 3o - 3u - 3o - 3u - 3o$.
2. $R \longrightarrow L$: $3o - 3u - 4o - 3u - 3o - u$.
3. $L \longrightarrow R$: $u - 3o - 3u - 4o - 3u - 3o$.
4. $R \longrightarrow L$: $u - 3o - 3u - 4o - 4u - 3o - u$.
5. $L \longrightarrow R$: $2u - 3o - 3u - 4o - 4u - 3o$.
6. $R \longrightarrow L$: $u - 4o - 3u - 4o - 4u - 4o - u$.
7. $L \longrightarrow R$: $2u - 4o - 3u - 4o - 4u - 4o$.
8. $R \longrightarrow L$: $u - 4o - 4u - 4o - 4u - 4o - u$.

After completion of 'Step 4', we are ready to start with 'Step 5', which is the interbraiding of the Turk's Head Knot with $s = 5$ parts. When this interbraiding is completed, we have five left bight-boundaries and five right bight-boundaries; hence $A = 5$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps $1, 2, 3, 4$ and 5, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 3 to the right bight-boundary 5, a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 2, a lower-left to upper-right half-cycle of the 'Step 3' Turk's Head Knot runs from the left bight-boundary 5 to the right bight-boundary 3, a lower-left to upper-right half-cycle of the 'Step 4' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 1, and a lower-left to upper-right half-cycle of the 'Step 5' Turk's Head Knot runs from the left bight-boundary 4 to the right bight-boundary 4. Hence for the 'Step 5' Turk's Head Knot the value for 'L' is 4 and the value for 'R' is 4, with $A = 5$ and $s = 5$.

The positions of the half-cycles from lower-left to upper-right belonging to the Steps $1, 2, 3, 4$ and 5 Turk's Head Knots are schematically illustrated, together with the values for L, R and A, in the lower left diagram of Fig. 60.

After substitution of $A = 5$, $L = 4$, $R = 4$ in the Algorithm-table for $B^* = 4$, $s = 5$ we obtain $L - 1 = 3$, $R - 1 = 3$, $A - 1 = 4$ and hence the table of Fig. 63 results.

$B^*=4$; *Enlarging cycle*



s=5

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	3	4	4	4	3
ref. R → L	3				3
1. L → R	3	4	4	4	3
2. R → L	3	4	4	5	3
3. L → R	3	4	4	5	3
4. R → L	3	4	5	5	3
5. L → R	3	4	5	5	3
6. R → L	3	5	5	5	3
7. L → R	3	5	5	5	3
8. R → L	4	5	5	5	3

Fig. 63 — Algorithm-tables for Step 5.

This table supplies us with the following braiding algorithm associated with 'Step 5':

1. $L \longrightarrow R$: $3u - 4o - 4u - 4o - 3u$.
2. $R \longrightarrow L$: $3u - 4o - 4u - 5o - 3u$.
3. $L \longrightarrow R$: $3u - 4o - 4u - 5o - 3u$.
4. $R \longrightarrow L$: $3u - 4o - 5u - 5o - 3u$.
5. $L \longrightarrow R$: $3u - 4o - 5u - 5o - 3u$.
6. $R \longrightarrow L$: $3u - 5o - 5u - 5o - 3u$.
7. $L \longrightarrow R$: $3u - 5o - 5u - 5o - 3u$.
8. $R \longrightarrow L$: $4u - 5o - 5u - 5o - 3u$.

Algorithm-tables

The remainder of this Chapter consists of Algorithm-tables, catering for B^*-values ranging from 3 to 7 inclusive, paired with s-values ranging from 3 to 13 inclusive.

The method for calculating the entries in the tables is explained in Chapter 7. Using this method, the reader may calculate Algorithm-tables for any desired pair of values for B^* and s.

As noted earlier (see page 37), the tables which now follow are generally given two to a page; the respective s-values are consecutive odd positive integers.

B*=3 ; *Enlarging cycle*

$$\begin{tikzcd}
L & \\
 & R \arrow[lu] \\
L \arrow[ru] &
\end{tikzcd}$$

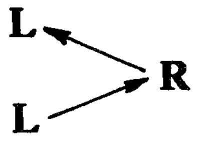
s=5

set number	1	2	3	4	5
crossing type	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1				L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A	A - 1	L - 1
3. L → R	L - 1	A - 1	A	A - 1	R - 1
4. R → L	R	A - 1	A	A	L - 1
5. L → R	L	A - 1	A	A	R - 1
6. R → L	R	A	A	A	L - 1

s=7

set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1						L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A	A	A - 1	A	A	L - 1
5. L → R	L - 1	A	A	A - 1	A	A	R - 1
6. R → L	R	A	A	A	A	A	L - 1

B* = 3 ; *Enlarging cycle*



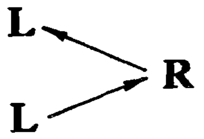
s = 11

set number	1	2	3	4	5	6	7	8	9	10	11
crossing type	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1										L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A	A - 1	A - 1	A	A - 1	A - 1	A	A - 1	L - 1
3. L → R	L - 1	A - 1	A	A - 1	A - 1	A	A - 1	A - 1	A	A - 1	R - 1
4. R → L	R	A - 1	A	A	A - 1	A	A	A - 1	A	A	L - 1
5. L → R	L	A - 1	A	A	A - 1	A	A	A - 1	A	A	R - 1
6. R → L	R	A	A	A	A	A	A	A	A	A	L - 1

s = 13

set number	1	2	3	4	5	6	7	8	9	10	11	12	13
crossing type	u	o	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1												L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A	A - 1	A - 1	A	A - 1	A - 1	A	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A	A - 1	A - 1	A	A - 1	A - 1	A	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A	A	A - 1	A	A	A - 1	A	A	A - 1	A	A	L - 1
5. L → R	L - 1	A	A	A - 1	A	A	A - 1	A	A	A - 1	A	A	R - 1
6. R → L	R	A	A	A	A	A	A	A	A	A	A	A	L - 1

$B^*=4$; *Enlarging cycle*



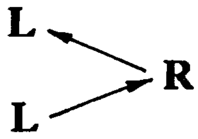
s=3

set number	1	2	3
crossing type	u	o	u
ref. L → R	L-1	A-1	R-1
ref. R → L	R-1		L-1
1. L → R	L-1	A-1	R-1
2. R → L	R-1	A-1	L-1
3. L → R	L-1	A-1	R-1
4. R → L	R	A-1	L-1
5. L → R	L	A-1	R-1
6. R → L	R	A	L-1
7. L → R	L	A	R-1
8. R → L	R	A	L-1

s=5

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	L-1	A-1	A-1	A-1	R-1
ref. R → L	R-1				L-1
1. L → R	L-1	A-1	A-1	A-1	R-1
2. R → L	R-1	A-1	A-1	A	L-1
3. L → R	L-1	A-1	A-1	A	R-1
4. R → L	R-1	A-1	A	A	L-1
5. L → R	L-1	A-1	A	A	R-1
6. R → L	R-1	A	A	A	L-1
7. L → R	L-1	A	A	A	R-1
8. R → L	R	A	A	A	L-1

B* =4 ; *Enlarging cycle*



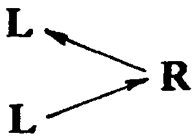
s=5

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1				L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A - 1	A	A	L - 1
5. L → R	L - 1	A - 1	A	A	R - 1
6. R → L	R - 1	A	A	A	L - 1
7. L → R	L - 1	A	A	A	R - 1
8. R → L	R	A	A	A	L - 1

s=7

set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1						L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A	A - 1	A - 1	R - 1
4. R → L	R	A - 1	A - 1	A	A	A - 1	L - 1
5. L → R	L	A - 1	A - 1	A	A	A - 1	R - 1
6. R → L	R	A	A - 1	A	A	A	L - 1
7. L → R	L	A	A - 1	A	A	A	R - 1
8. R → L	R	A	A	A	A	A	L - 1

$B^* = 4$; *Enlarging cycle*



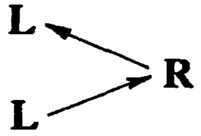
s = 7

set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1						L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A	A - 1	A - 1	R - 1
4. R → L	R	A - 1	A - 1	A	A	A - 1	L - 1
5. L → R	L	A - 1	A - 1	A	A	A - 1	R - 1
6. R → L	R	A	A - 1	A	A	A	L - 1
7. L → R	L	A	A - 1	A	A	A	R - 1
8. R → L	R	A	A	A	A	A	L - 1

s = 9

set number	1	2	3	4	5	6	7	8	9
crossing type	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1								L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A - 1	A	A	A - 1	A - 1	A	A	L - 1
5. L → R	L - 1	A - 1	A	A	A - 1	A - 1	A	A	R - 1
6. R → L	R - 1	A	A	A	A - 1	A	A	A	L - 1
7. L → R	L - 1	A	A	A	A - 1	A	A	A	R - 1
8. R → L	R	A	A	A	A	A	A	A	L - 1

$B^*=4$; *Enlarging cycle*



s=9

set number	1	2	3	4	5	6	7	8	9
crossing type	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1								L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A - 1	A	A	A - 1	A - 1	A	A	L - 1
5. L → R	L - 1	A - 1	A	A	A - 1	A - 1	A	A	R - 1
6. R → L	R - 1	A	A	A	A - 1	A	A	A	L - 1
7. L → R	L - 1	A	A	A	A - 1	A	A	A	R - 1
8. R → L	R	A	A	A	A	A	A	A	L - 1

s=11

set number	1	2	3	4	5	6	7	8	9	10	11
crossing type	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1										L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	R - 1
4. R → L	R	A - 1	A - 1	A	A	A - 1	A - 1	A	A	A - 1	L - 1
5. L → R	L	A - 1	A - 1	A	A	A - 1	A - 1	A	A	A - 1	R - 1
6. R → L	R	A	A - 1	A	A	A	A - 1	A	A	A	L - 1
7. L → R	L	A	A - 1	A	A	A	A - 1	A	A	A	R - 1
8. R → L	R	A	A	A	A	A	A	A	A	A	L - 1

$B^* = 4$; *Enlarging cycle*

$$\begin{tikzcd} L & \\ & R \arrow[ul] \\ L \arrow[ur] & \end{tikzcd}$$

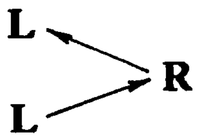
s=11

set number	1	2	3	4	5	6	7	8	9	10	11
crossing type	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1										L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	R - 1
4. R → L	R	A - 1	A - 1	A	A	A - 1	A - 1	A	A	A - 1	L - 1
5. L → R	L	A - 1	A - 1	A	A	A - 1	A - 1	A	A	A - 1	R - 1
6. R → L	R	A	A - 1	A	A	A	A - 1	A	A	A	L - 1
7. L → R	L	A	A - 1	A	A	A	A - 1	A	A	A	R - 1
8. R → L	R	A	A	A	A	A	A	A	A	A	L - 1

s=13

set number	1	2	3	4	5	6	7	8	9	10	11	12	13
crossing type	u	o	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1												L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A - 1	A	A	A - 1	A - 1	A	A	A - 1	A - 1	A	A	L - 1
5. L → R	L - 1	A - 1	A	A	A - 1	A - 1	A	A	A - 1	A - 1	A	A	R - 1
6. R → L	R - 1	A	A	A	A - 1	A	A	A	A - 1	A	A	A	L - 1
7. L → R	L - 1	A	A	A	A - 1	A	A	A	A - 1	A	A	A	R - 1
8. R → L	R	A	A	A	A	A	A	A	A	A	A	A	L - 1

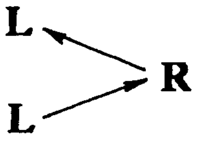
$B^*=5$; *Enlarging cycle*



s=3

set number	1	2	3
crossing type	u	o	u
ref. L → R	L - 1	A - 1	R - 1
ref. R → L	R - 1		L - 1
1. L → R	L - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	R - 1
4. R → L	R - 1	A	L - 1
5. L → R	L - 1	A	R - 1
6. R → L	R - 1	A	L - 1
7. L → R	L - 1	A	R - 1
8. R → L	R	A	L - 1
9. L → R	L	A	R - 1
10. R → L	R	A	L - 1

$B^* = 5$; *Enlarging cycle*



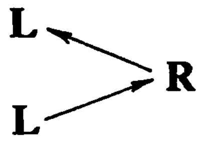
s=7

set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1						L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A	A - 1	R - 1
4. R → L	R - 1	A - 1	A	A - 1	A	A - 1	L - 1
5. L → R	L - 1	A - 1	A	A - 1	A	A - 1	R - 1
6. R → L	R	A - 1	A	A - 1	A	A	L - 1
7. L → R	L	A - 1	A	A - 1	A	A	R - 1
8. R → L	R	A - 1	A	A	A	A	L - 1
9. L → R	L	A - 1	A	A	A	A	R - 1
10. R → L	R	A	A	A	A	A	L - 1

s=9

set number	1	2	3	4	5	6	7	8	9
crossing type	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1								L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	R - 1
4. R → L	R	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	L - 1
5. L → R	L	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	R - 1
6. R → L	R	A	A - 1	A - 1	A	A	A	A - 1	L - 1
7. L → R	L	A	A - 1	A - 1	A	A	A	A - 1	R - 1
8. R → L	R	A	A	A - 1	A	A	A	A	L - 1
9. L → R	L	A	A	A - 1	A	A	A	A	R - 1
10. R → L	R	A	A	A	A	A	A	A	L - 1

$B^*=5$; *Enlarging cycle*



s=9

set number	1	2	3	4	5	6	7	8	9
crossing type	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1								L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	R - 1
4. R → L	R	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	L - 1
5. L → R	L	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	R - 1
6. R → L	R	A	A - 1	A - 1	A	A	A	A - 1	L - 1
7. L → R	L	A	A - 1	A - 1	A	A	A	A - 1	R - 1
8. R → L	R	A	A	A - 1	A	A	A	A	L - 1
9. L → R	L	A	A	A - 1	A	A	A	A	R - 1
10. R → L	R	A	A	A	A	A	A	A	L - 1

s=11

set number	1	2	3	4	5	6	7	8	9	10	11
crossing type	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1										L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	A	A	L - 1
5. L → R	L - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	A	A	R - 1
6. R → L	R - 1	A - 1	A	A	A	A - 1	A - 1	A	A	A	L - 1
7. L → R	L - 1	A - 1	A	A	A	A - 1	A - 1	A	A	A	R - 1
8. R → L	R - 1	A	A	A	A	A - 1	A	A	A	A	L - 1
9. L → R	L - 1	A	A	A	A	A - 1	A	A	A	A	R - 1
10. R → L	R	A	A	A	A	A	A	A	A	A	L - 1

$B^*=5$; *Enlarging cycle*

$$\begin{tikzcd} L & \\ & R \arrow[ul] \\ L \arrow[ur] & \end{tikzcd}$$

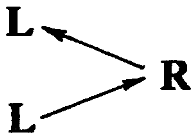
s=11

set number	1	2	3	4	5	6	7	8	9	10	11
crossing type	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1										L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	A	A	L - 1
5. L → R	L - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	A	A	R - 1
6. R → L	R - 1	A - 1	A	A	A	A - 1	A - 1	A	A	A	L - 1
7. L → R	L - 1	A - 1	A	A	A	A - 1	A - 1	A	A	A	R - 1
8. R → L	R - 1	A	A	A	A	A - 1	A	A	A	A	L - 1
9. L → R	L - 1	A	A	A	A	A - 1	A	A	A	A	R - 1
10. R → L	R	A	A	A	A	A	A	A	A	A	L - 1

s=13

set number	1	2	3	4	5	6	7	8	9	10	11	12	13
crossing type	u	o	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1												L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	R - 1
4. R → L	R - 1	A	A - 1	A - 1	A	A - 1	A	A - 1	A - 1	A	A - 1	A	L - 1
5. L → R	L - 1	A	A - 1	A - 1	A	A - 1	A	A - 1	A - 1	A	A - 1	A	R - 1
6. R → L	R - 1	A	A - 1	A	A	A - 1	A	A - 1	A	A	A - 1	A	L - 1
7. L → R	L - 1	A	A - 1	A	A	A - 1	A	A - 1	A	A	A - 1	A	R - 1
8. R → L	R	A	A - 1	A	A	A	A	A - 1	A	A	A	A	L - 1
9. L → R	L	A	A - 1	A	A	A	A	A - 1	A	A	A	A	R - 1
10. R → L	R	A	A	A	A	A	A	A	A	A	A	A	L - 1

$B^*=6$; *Enlarging cycle*



s=5

set number	1	2	3	4	5
crossing type	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1				L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
4. R → L	R	A - 1	A - 1	A - 1	L - 1
5. L → R	L	A - 1	A - 1	A - 1	R - 1
6. R → L	R	A	A - 1	A - 1	L - 1
7. L → R	L	A	A - 1	A - 1	R - 1
8. R → L	R	A	A	A - 1	L - 1
9. L → R	L	A	A	A - 1	R - 1
10. R → L	R	A	A	A	L - 1
11. L → R	L	A	A	A	R - 1
12. R → L	R	A	A	A	L - 1

s=7

set number	1	2	3	4	5	6	7
crossing type	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1						L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A - 1	A - 1	A - 1	A	A	L - 1
5. L → R	L - 1	A - 1	A - 1	A - 1	A	A	R - 1
6. R → L	R - 1	A - 1	A - 1	A	A	A	L - 1
7. L → R	L - 1	A - 1	A - 1	A	A	A	R - 1
8. R → L	R - 1	A - 1	A	A	A	A	L - 1
9. L → R	L - 1	A - 1	A	A	A	A	R - 1
10. R → L	R - 1	A	A	A	A	A	L - 1
11. L → R	L - 1	A	A	A	A	A	R - 1
12. R → L	R	A	A	A	A	A	L - 1

$B^*=6$; *Enlarging cycle*

$$\begin{tikzcd} L & \\ & R \arrow[ul] \\ L \arrow[ur] & \end{tikzcd}$$

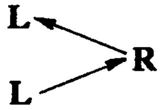
s = 11

set number	1	2	3	4	5	6	7	8	9	10	11
crossing type	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1										L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	R - 1
4. R → L	R	A - 1	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	L - 1
5. L → R	L	A - 1	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	R - 1
6. R → L	R	A	A - 1	A - 1	A - 1	A	A	A	A - 1	A - 1	L - 1
7. L → R	L	A	A - 1	A - 1	A - 1	A	A	A	A - 1	A - 1	R - 1
8. R → L	R	A	A	A - 1	A - 1	A	A	A	A	A - 1	L - 1
9. L → R	L	A	A	A - 1	A - 1	A	A	A	A	A - 1	R - 1
10. R → L	R	A	A	A	A - 1	A	A	A	A	A	L - 1
11. L → R	L	A	A	A	A - 1	A	A	A	A	A	R - 1
12. R → L	R	A	A	A	A	A	A	A	A	A	L - 1

s = 13

set number	1	2	3	4	5	6	7	8	9	10	11	12	13
crossing type	u	o	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1												L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A - 1	A	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A - 1	A	R - 1
4. R → L	R - 1	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	A - 1	A	A	L - 1
5. L → R	L - 1	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	A - 1	A	A	R - 1
6. R → L	R - 1	A - 1	A - 1	A	A	A	A - 1	A - 1	A - 1	A	A	A	L - 1
7. L → R	L - 1	A - 1	A - 1	A	A	A	A - 1	A - 1	A - 1	A	A	A	R - 1
8. R → L	R - 1	A - 1	A	A	A	A	A - 1	A - 1	A	A	A	A	L - 1
9. L → R	L - 1	A - 1	A	A	A	A	A - 1	A - 1	A	A	A	A	R - 1
10. R → L	R - 1	A	A	A	A	A	A - 1	A	A	A	A	A	L - 1
11. L → R	L - 1	A	A	A	A	A	A - 1	A	A	A	A	A	R - 1
12. R → L	R	A	A	A	A	A	A	A	A	A	A	A	L - 1

$B^*=7$; *Enlarging cycle*



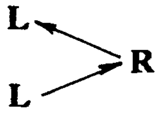
s=3

set number	1	2	3
crossing type	u	o	u
ref. L → R	L - 1	A - 1	R - 1
ref. R → L	R - 1		L - 1
1. L → R	L - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	R - 1
4. R → L	R - 1	A - 1	L - 1
5. L → R	L - 1	A - 1	R - 1
6. R → L	R	A - 1	L - 1
7. L → R	L	A - 1	R - 1
8. R → L	R	A - 1	L - 1
9. L → R	L	A - 1	R - 1
10. R → L	R	A	L - 1
11. L → R	L	A	R - 1
12. R → L	R	A	L - 1
13. L → R	L	A	R - 1
14. R → L	R	A	L - 1

s=5

set number	1	2	3	4	3
crossing type	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1				L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	R - 1
4. R → L	R - 1	A	A - 1	A - 1	L - 1
5. L → R	L - 1	A	A - 1	A - 1	R - 1
6. R → L	R - 1	A	A - 1	A	L - 1
7. L → R	L - 1	A	A - 1	A	R - 1
8. R → L	R - 1	A	A - 1	A	L - 1
9. L → R	L - 1	A	A - 1	A	R - 1
10. R → L	R	A	A - 1	A	L - 1
11. L → R	L	A	A - 1	A	R - 1
12. R → L	R	A	A	A	L - 1
13. L → R	L	A	A	A	R - 1
14. R → L	R	A	A	A	L - 1

$B^*=7$; *Enlarging cycle*



s=9

set number	1	2	3	4	5	6	7	8	9
crossing type	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1								L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	R - 1
4. R → L	R - 1	A - 1	A - 1	A - 1	A	A - 1	A	A - 1	L - 1
5. L → R	L - 1	A - 1	A - 1	A - 1	A	A - 1	A	A - 1	R - 1
6. R → L	R - 1	A - 1	A	A - 1	A	A - 1	A	A - 1	L - 1
7. L → R	L - 1	A - 1	A	A - 1	A	A - 1	A	A - 1	R - 1
8. R → L	R	A - 1	A	A - 1	A	A - 1	A	A	L - 1
9. L → R	L	A - 1	A	A - 1	A	A - 1	A	A	R - 1
10. R → L	R	A - 1	A	A - 1	A	A	A	A	L - 1
11. L → R	L	A - 1	A	A - 1	A	A	A	A	R - 1
12. R → L	R	A - 1	A	A	A	A	A	A	L - 1
13. L → R	L	A - 1	A	A	A	A	A	A	R - 1
14. R → L	R	A	A	A	A	A	A	A	L - 1

s=11

set number	1	2	3	4	5	6	7	8	9	10	11
crossing type	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1										L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	R - 1
4. R → L	R - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	A	L - 1
5. L → R	L - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	A	R - 1
6. R → L	R - 1	A - 1	A	A - 1	A - 1	A	A	A - 1	A - 1	A	L - 1
7. L → R	L - 1	A - 1	A	A - 1	A - 1	A	A	A - 1	A - 1	A	R - 1
8. R → L	R - 1	A	A	A - 1	A - 1	A	A	A - 1	A	A	L - 1
9. L → R	L - 1	A	A	A - 1	A - 1	A	A	A - 1	A	A	R - 1
10. R → L	R - 1	A	A	A - 1	A	A	A	A - 1	A	A	L - 1
11. L → R	L - 1	A	A	A - 1	A	A	A	A - 1	A	A	R - 1
12. R → L	R	A	A	A - 1	A	A	A	A	A	A	L - 1
13. L → R	L	A	A	A - 1	A	A	A	A	A	A	R - 1
14. R → L	R	A	A	A	A	A	A	A	A	A	L - 1

B* = 7 ; *Enlarging cycle* $\begin{tikzcd} L & \\ & R \arrow[ul] \\ L \arrow[ur] & \end{tikzcd}$

s = 11

set number	1	2	3	4	5	6	7	8	9	10	11
crossing type	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1										L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	R - 1
4. R → L	R - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	A	L - 1
5. L → R	L - 1	A - 1	A	A - 1	A - 1	A - 1	A	A - 1	A - 1	A	R - 1
6. R → L	R - 1	A - 1	A	A - 1	A - 1	A	A	A - 1	A - 1	A	L - 1
7. L → R	L - 1	A - 1	A	A - 1	A - 1	A	A	A - 1	A - 1	A	R - 1
8. R → L	R - 1	A	A	A - 1	A - 1	A	A	A - 1	A	A	L - 1
9. L → R	L - 1	A	A	A - 1	A - 1	A	A	A - 1	A	A	R - 1
10. R → L	R - 1	A	A	A - 1	A	A	A	A - 1	A	A	L - 1
11. L → R	L - 1	A	A	A - 1	A	A	A	A - 1	A	A	R - 1
12. R → L	R	A	A	A - 1	A	A	A	A	A	A	L - 1
13. L → R	L	A	A	A - 1	A	A	A	A	A	A	R - 1
14. R → L	R	A	A	A	A	A	A	A	A	A	L - 1

s = 13

set number	1	2	3	4	5	6	7	8	9	10	11	12	13
crossing type	u	o	u	o	u	o	u	o	u	o	u	o	u
ref. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
ref. R → L	R - 1												L - 1
1. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
2. R → L	R - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A - 1	L - 1
3. L → R	L - 1	A - 1	A - 1	A - 1	A - 1	A - 1	A	A - 1	A - 1	A - 1	A - 1	A - 1	R - 1
4. R → L	R	A - 1	A - 1	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	A - 1	L - 1
5. L → R	L	A - 1	A - 1	A - 1	A - 1	A - 1	A	A	A - 1	A - 1	A - 1	A - 1	R - 1
6. R → L	R	A	A - 1	A - 1	A - 1	A - 1	A	A	A	A - 1	A - 1	A - 1	L - 1
7. L → R	L	A	A - 1	A - 1	A - 1	A - 1	A	A	A	A - 1	A - 1	A - 1	R - 1
8. R → L	R	A	A	A - 1	A - 1	A - 1	A	A	A	A	A - 1	A - 1	L - 1
9. L → R	L	A	A	A - 1	A - 1	A - 1	A	A	A	A	A - 1	A - 1	R - 1
10. R → L	R	A	A	A	A - 1	A - 1	A	A	A	A	A	A - 1	L - 1
11. L → R	L	A	A	A	A - 1	A - 1	A	A	A	A	A	A - 1	R - 1
12. R → L	R	A	A	A	A	A - 1	A	A	A	A	A	A	L - 1
13. L → R	L	A	A	A	A	A - 1	A	A	A	A	A	A	R - 1
14. R → L	R	A	A	A	A	A	A	A	A	A	A	A	L - 1

6

CLASSIFYING THE STANDARD HERRINGBONE PINEAPPLE KNOTS INTO TYPES

In an A-pass Standard Herringbone Pineapple Knot, a lower-left to upper-right half-cycle starting from the left bight-boundary **1** can either run to the right bight-boundary **1** or **2** or **3** or $\cdots$ or $\mathbf{A-1}$ or $\mathbf{A}$. We note that the right bight-boundary value is equal to the value of a_2 when $a_2 > 0$ and is equal to A when $a_2 = 0$. See page 38 for verification of this.

The position of every half-cycle in a Standard Herringbone Pineapple Knot (relative to the left and right bight-boundaries) is determined as soon as the position of the lower-left to upper-right half-cycle starting at the left bight-boundary **1** is known. Hence we can classify the Standard Herringbone Pineapple Knots with A passes into A Types, whereby the Type-number is given by the a_2-value when $a_2 > 0$ and by the A-value when $a_2 = 0$.

We shall use the notation $\mathbf{S_1}$ to denote the Type of Standard Herringbone Pineapple Knot which has a half-cycle running from the left bight-boundary **1** to the right bight-boundary **1**; similarly $\mathbf{S_2}$ denotes Standard Herringbone Pineapple Knots which have a half-cycle running from the left bight-boundary **1** to the right bight-boundary **2**; and so on.

Hence in general $\mathbf{S_{a_2}}$ denotes Standard Herringbone Pineapple Knots which have a half-cycle running from the left bight-boundary **1** to the right bight-boundary $\mathbf{a_2}$. The Type associated with $\mathbf{a_2 = 0}$ is denoted by $\mathbf{S_A}$.

Looking back at our examples of chapter 5, we see that the Standard Herringbone Pineapple Knot of example 1 is of Type $\mathbf{S_1}$, whereas the Standard Herringbone Pineapple Knot of example 2 is of Type $\mathbf{S_2}$.

Several methods are in use for braiding Standard Herringbone Pineapple Knots. Most of these methods involve the memorising of 'sequence patterns' in the braid. These methods may be fine for those who regularly braid only a very small variety of Standard Herringbone Pineapple Knots; however, the serious braider is strongly advised against using such restrictive methods, since they invariably lead, sooner or later, to misconceptions and erroneous conclusions. For example, there exists literature which deals only with the two Types (that is, $\mathbf{S_1}$ and $\mathbf{S_2}$) of the 2-pass Standard Herringbone Pineapple Knots; and it wrongly refers to four Types, when there are only two. There are, indeed, in case of 2-pass Standard Herringbone Pineapple Knots, two ways to braid each Type, so there are four methods of braiding them; but only two different Types of these Knots result. (For instance in Example 1 of Chapter 5, it is immaterial whether we use the braiding method (i) or the braiding method (ii).) This is the kind of misconception that can arise when a very restricted set of members of an infinite family is studied or written about.

Moreover, it is of the utmost importance that the braider fully understands all the properties of the general A-pass Standard Herringbone Pineapple Knot and the $A!$ ways in which each of the A Types can be braided, otherwise the braider may draw erroneous conclusions. An example of this is to be found in the *'Encyclopedia of Rawhide and Leather Braiding'* by Bruce Grant:

On page 418 we find the *'Bruce Knot'* and the *'Catherine Knot'*. Both these knots are Standard Herringbone Pineapple Knots, and hence do not deserve a special name any more than would any other Standard Herringbone Pineapple Knot.

The *'Bruce Knot'* is a 2-pass Standard Herringbone Pineapple Knot with $a_2 = 0$ (hence $a_1 = 2$), $2m - 1 = 3$, $x = 4$, $P = 6$ and $B^* = 5$. Since $a_2 = 0$, the Knot is of Type S_2 .

The *'Catherine Knot'* is also a 2-pass Standard Herringbone Pineapple Knot with $a_1 = 1$, $a_2 = 1$, $2m - 1 = 3$, $2m + 1 = 5$, $x = 6$, $P = 8$ and $B^* = 4$. Since $a_2 = 1$, the Knot is of Type S_1 .

The *'Bruce Knot'* is braided as follows:

Step 1. — Braid the component with half-cycle $1 \longrightarrow 2$.
Step 2. — Braid the component with half-cycle $2 \longrightarrow 1$.

The component in 'Step 1' is the foundation Turk's Head Knot and the component in 'Step 2' is the interwoven Turk's Head Knot.

When 'Step 1' is completed, the result is a Turk's head Knot with 5-bights and 3-parts. This knot, the foundation Turk's Head Knot, can be regarded as a Standard Herringbone Pineapple Knot with $A = 1$ and $B^* = 5$. Since $A = 1$ the left bight-boundary as well as the right bight-boundary of this knot carries the number 1; hence in the Algorithm-table which will supply us with the braiding algorithm for this knot, the value of 'L' is 1 and the value of 'R' is 1. Furthermore the value of 'A' is 1; the value of 'B^*' is 5 and the value of 's' is 3.

After substitution of $A = 1$, $L = 1$, $R = 1$ in the Algorithm-table for $B^* = 5$, $s = 3$ we obtain $L - 1 = 0$, $R - 1 = 0$, $A - 1 = 0$, and hence this table supplies us with the following braiding algorithm associated with 'Step 1':

1. $L \longrightarrow R$: free run.
2. $R \longrightarrow L$: free run.
3. $L \longrightarrow R$: free run.
4. $R \longrightarrow L$: o.
5. $L \longrightarrow R$: o.
6. $R \longrightarrow L$: o.
7. $L \longrightarrow R$: o.
8. $R \longrightarrow L$: $u - o$.
9. $L \longrightarrow R$: $u - o$.
10. $R \longrightarrow L$: $u - o$.

After completion of 'Step 1', we are ready to start with 'Step 2': the interbraiding of the Turk's Head Knot with $s = 3$ parts. When this interbraiding is completed, we have two left bight-boundaries and two right bight-boundaries; hence $A = 2$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1 and 2, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 2, and a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 1. Hence for the 'Step 2' Turk's Head Knot the value for 'L' is 2 and the value for 'R' is 1, with $A = 2$ and $s = 3$.

After substitution of $A = 2$, $L = 2$, $R = 1$ in the Algorithm-table for $B^* = 5$, $s = 3$ we obtain $L - 1 = 1$, $R - 1 = 0$, $A - 1 = 1$ and hence this table supplies us with the following braiding algorithm associated with 'Step 2':

1. $L \longrightarrow R$: $u - o$.
2. $R \longrightarrow L$: $o - u$.
3. $L \longrightarrow R$: $u - o$.
4. $R \longrightarrow L$: $2o - u$.
5. $L \longrightarrow R$: $u - 2o$.
6. $R \longrightarrow L$: $2o - u$.
7. $L \longrightarrow R$: $u - 2o$.
8. $R \longrightarrow L$: $u - 2o - u$.
9. $L \longrightarrow R$: $2u - 2o$.
10. $R \longrightarrow L$: $u - 2o - u$.

We shall now give the braiding of the *'Catherine Knot'*:

Step 1. — Braid the component with half-cycle $2 \longrightarrow 2$.
Step 2. — Braid the component with half-cycle $1 \longrightarrow 1$.

The component in 'Step 1' is the foundation Turk's Head Knot and the component in 'Step 2' is the interwoven Turk's Head Knot.

When 'Step 1' is completed, the result is a Turk's Head Knot with 4-bights and 3-parts. This knot, the foundation Turk's Head Knot, can be regarded as a Standard Herringbone Pineapple Knot with $A = 1$ and $B^* = 4$. Since $A = 1$ the left bight-boundary as well as the right bight-boundary of this knot carries the number 1; hence in the table which will supply us with the braiding algorithm for this knot the value of 'L' is 1 and the value of 'R' is 1. Furthermore the value of 'A' is 1; the value of 'B^*' is 4 and the value of 's' is 3.

After substitution of $A = 1$, $L = 1$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 3$ we obtain $L - 1 = 0$, $R - 1 = 0$, $A - 1 = 0$ and hence this table supplies us with the following braiding algorithm associated with 'Step 1':

1. $L \longrightarrow R$: free run.
2. $R \longrightarrow L$: free run.
3. $L \longrightarrow R$: free run.
4. $R \longrightarrow L$: u.
5. $L \longrightarrow R$: u.
6. $R \longrightarrow L$: $u - o$.
7. $L \longrightarrow R$: $u - o$.
8. $R \longrightarrow L$: $u - o$.

After completion of 'Step 1', we are ready to start with 'Step 2', which is the interbraiding of the Turk's Head Knot with $s = 5$ parts. When this interbraiding is completed, we have two left bight-boundaries and two right bight-boundaries; hence $A = 2$. In the grid-diagram belonging to the Standard Herringbone Pineapple Knot which is obtained after the completion of the Steps 1 and 2, a lower-left to upper-right half-cycle of the 'Step 1' Turk's Head Knot runs from the left bight-boundary 2 to the right bight-boundary 2, and a lower-left to upper-right half-cycle of the 'Step 2' Turk's Head Knot runs from the left bight-boundary 1 to the right bight-boundary 1. Hence for the 'Step 2' Turk's Head Knot the value for 'L' is 1 and the value for 'R' is 1, with $A = 2$ and $s = 5$.

After substitution of $A = 2$, $L = 1$, $R = 1$ in the Algorithm-table for $B^* = 4$, $s = 5$ we obtain $L - 1 = 0$, $R - 1 = 0$, $A - 1 = 1$ and hence this table supplies us with the following braiding algorithm associated with 'Step 2':

1. $L \longrightarrow R$: $o - u - o$.
2. $R \longrightarrow L$: $o - u - 2o$.
3. $L \longrightarrow R$: $o - u - 2o$.
4. $R \longrightarrow L$: $o - 2u - 2o$.
5. $L \longrightarrow R$: $o - 2u - 2o$.
6. $R \longrightarrow L$: $2o - 2u - 2o$.
7. $L \longrightarrow R$: $2o - 2u - 2o$.
8. $R \longrightarrow L$: $u - 2o - 2u - 2o$.

7

THE CALCULATION OF THE ENTRIES IN THE ALGORITHM-TABLES

At the beginning of Chapter 5 we have explained the general layout of the Algorithm-tables. We shall now explain how the entries for the body of an Algorithm-table are obtained. We have shown that the **reference** algorithm for any **odd** half-cycle (a half-cycle from lower-left to upper-right) is given by:

$(L-1)u-(A-1)o-(A-1)u-(A-1)o-(A-1)u-\cdots-(A-1)o-(A-1)u-(A-1)o-(R-1)u$.

Furthermore we have shown that the **reference** algorithm for any **even** half-cycle (a half-cycle from lower-right to upper-left) is given by:

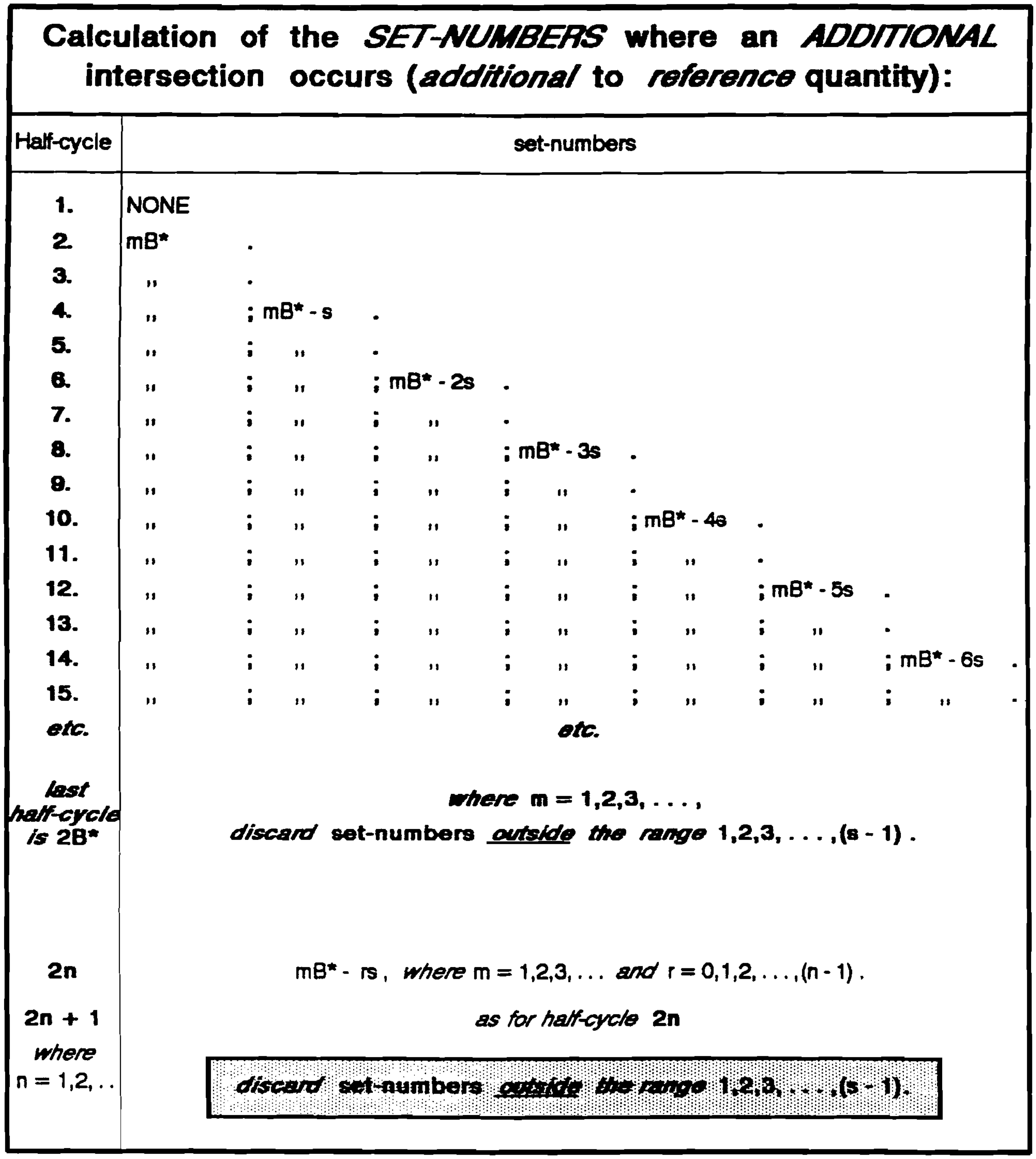
$(R-1)u-(A-1)o-(A-1)u-(A-1)o-(A-1)u-\cdots-(A-1)o-(A-1)u-(A-1)o-(L-1)u$.

The algorithm for the **first half-cycle** is always identical to the **reference** algorithm for an **odd** half-cycle.

For other half-cycles the reference quantities $(L-1)$, $(R-1)$ may have increased by 1 for some entries associated with **set-number** 1, and the quantity $(A-1)$ may have increased by 1 for some set-numbers. Hence these quantities have become respectively L , R and A . In the last column, no such change occurs.

The set-numbers for which such an increase occurs, can readily be calculated by means of the table in Fig. 64. This table works as follows:

The 1st half-cycle indicates '**NONE**'. This means that there is no additional intersection (hence no increase of 1 to the reference quantity) for any of the set-numbers; in other words, the algorithm for the 1st half-cycle is identical to the reference algorithm of an odd half-cycle, as we have already mentioned above.



Calculation of the *SET-NUMBERS* where an *ADDITIONAL* intersection occurs (*additional* to *reference* quantity):

Half-cycle	set-numbers						
1.	NONE						
2.	mB* .						
3.	" .						
4.	"	; mB* - s .					
5.	"	; " .					
6.	"	; "	; mB* - 2s .				
7.	"	; "	; " .				
8.	"	; "	; "	; mB* - 3s .			
9.	"	; "	; "	; " .			
10.	"	; "	; "	; "	; mB* - 4s .		
11.	"	; "	; "	; "	; " .		
12.	"	; "	; "	; "	; "	; mB* - 5s .	
13.	"	; "	; "	; "	; "	; " .	
14.	"	; "	; "	; "	; "	; "	; mB* - 6s .
15.	"	; "	; "	; "	; "	; "	; " .
etc.	*etc.*						
last half-cycle is 2B*	***where*** **m = 1,2,3, . . . ,** *discard* set-numbers ***outside the range*** 1,2,3, . . . ,(s - 1) .						
2n	mB* - rs , *where* m = 1,2,3, . . . *and* r = 0,1,2, . . . ,(n - 1) .						
2n + 1 *where* n = 1,2, . .	*as for half-cycle* 2n ***discard*** **set-numbers *outside the range* 1,2,3, . . . ,(s - 1).**						

Fig. 64 — Calculation-table for additional set-number intersections.

The 2^{nd} half-cycle gives mB^*. The table indicates that the value of m may be $1, 2, 3, \cdots$; furthermore, that set-number values **outside** the range $1, 2, 3, \cdots, (s-1)$ have to be discarded. Suppose $s = 13$ and $B^* = 4$ (the reader is reminded that the values for s and B^* **must** be coprime), then valid values for mB^* are $4, 8, 12$. Thus the reference entries for the set-numbers $4, 8$ and 12 increase by 1. Hence the entries for the set-numbers $4, 8$ and 12 in the algorithm table for $B^* = 4$ and $s = 13$, for the 2^{nd} half-cycle, are A.

The set-numbers of an **odd** half-cycle for which the reference quantity increases by 1 have the same value as the set-numbers of the **immediately preceding even** half-cycle for which the reference quantity increases by 1. Hence in our example, the entries for the set-numbers $4, 8$ and 12 in the algorithm table for $B^* = 4$ and $s = 13$, for the 3^{rd} half-cycle, are A.

The 4^{th} half-cycle gives mB^* and $(mB^* - s)$. Hence in **addition** to the set-numbers already found for mB^* (the set-numbers $4, 8, 12$), we obtain set-numbers for $(mB^* - s)$. Thus in our example we obtain the following further set-numbers 3 $((4 \times 4) - 13 = 3)$, 7 $((5 \times 4) - 13 = 7)$, and 11 $((6 \times 4) - 13 = 11)$. Consequently the entries for the set-numbers $3, 4, 7, 8, 11$ and 12 in the algorithm table for $B^* = 4$ and $s = 13$, for the 4^{th} half-cycle, are A.

The set-numbers of the 5^{th} half-cycle for which the reference quantity increases by 1 have the same value as the set-numbers of the **immediately preceding even** half-cycle 4.

The 6^{th} half-cycle gives mB^*, $(mB^* - s)$ and $(mB^* - 2s)$. Hence in **addition** to the set-numbers we already found for mB^* (the set-numbers $4, 8, 12$) and for $(mB^* - s)$ (the set-numbers $3, 7, 11$), we obtain set-numbers for $(mB^* - 2s)$. Thus in our example we obtain the further set-numbers 2 $((7 \times 4) - (2 \times 13) = 2)$, 6 $((8 \times 4) - (2 \times 13) = 6)$, and 10 $((9 \times 4) - (2 \times 13) = 10)$. Consequently the entries for the set-numbers $2, 3, 4, 6, 7, 8, 10, 11$ and 12 in the algorithm table for $B^* = 4$ and $s = 13$, for the 6^{th} half-cycle, are A.

And so on.

The reader should now be able to draw up algorithm tables additional to the ones supplied in this book, when the need for such larger tables arises.

We would like to advise the more serious braider who desires to obtain a greater understanding of relationships between braiding processes to study the contents of Appendix 4, where we show the close relationship between the table of Fig. 64 and the calculation of braiding algorithms for the Regular Knots.

Appendix 1

THE REGULAR-NESTED CYLINDRICAL BRAIDS

This Appendix is a summary of the properties of the six Regular-Nested Cylindrical Braid sub-classes, all of which belong to the class of Regular-Nested Cylindrical Braids.

Regular-Nested Cylindrical Braids
1) Standard Regular-Nested Cylindrical Braids.
2) Semi-Standard Regular-Nested Cylindrical Braids.
3) Perfect Regular-Nested Cylindrical Braids.
4) Semi-Perfect Regular-Nested Cylindrical Braids.
5) Compound Regular-Nested Cylindrical Braids.
6) Semi-Compound Regular-Nested Cylindrical Braids.

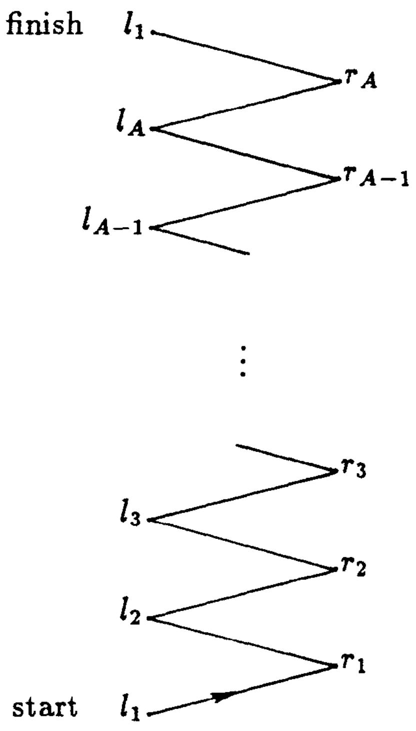
Each of these sub-classes is determined by a set of string-run properties, which are expressed by formulae given in this Appendix. The reader should note that the classifications are independent of coding. Any type of coding can be applied to any member of these sub-classes; and weaving algorithms can then be computed for constructing them.

Braids can be braided in two ways, according as their cycles are laid down from lower-left to upper-left (upwards), or from upper-left to lower-left (downwards). The formulae which express their string-run characteristics depend on which of these ways is used. Therefore we give the formulae in two Sections, I and II. Section I deals with the case that braiding is upwards; whereas Section II treats the case where braiding is downwards.

The examples used in Section II match the corresponding ones in Section I.

(I)

THE FORMULAE FOR BRAIDING UPWARDS



$$
\begin{array}{ll}
l_2 = |l_1 + \Delta|_A & r_2 = |r_1 - \Delta|_A \\
l_3 = |l_2 + \Delta|_A & r_3 = |r_2 - \Delta|_A \\
l_4 = |l_3 + \Delta|_A & r_4 = |r_3 - \Delta|_A \\
\vdots & \vdots \\
l_{i+1} = |l_i + \Delta|_A & r_{i+1} = |r_i - \Delta|_A \\
\vdots & \vdots
\end{array}
$$

In the above, $i = 1, 2, 3, \dots, (A-1)$ and
Δ is an integer belonging to the set: $\{0, 1, 2, 3, \dots, (A-2), (A-1)\}$.

All components of a given braid have the same Δ-value.

$\delta = |2(l_i + r_i) + \Delta|_A$ where δ is an integer belonging to the set: $\{1, 2, 3, \dots, (A-1), A\}$.

All components of a given braid have the same δ-value.

For all half-cycles of a given braid, the value of $(l_i + r_i)$ is either equal to:
(1) $A + 1$
or
(2) Y, or $Y + A$, where $2 \leq Y \leq A$.

All components of a given braid have the same x-value, and since $x = cA + \delta$ it follows that all components of a given braid have the same c-value.

In general we have:

$$
\begin{array}{lll}
\text{Type 1:} & \Delta_1 = \epsilon & \text{for } 1 \leq \epsilon < A/2\,, \\
\text{Type 2:} & \Delta_2 = A - \epsilon & \text{for } 1 \leq \epsilon < A/2\,.
\end{array}
$$

$$\text{Thus} \quad |l_{i_2} + r_{i_2}|_A = |l_{i_1} + r_{i_1} + \Delta_1|_A\,.$$

Note that Type 1 and Type 2 have the same Δ-value when $\Delta_1 = \Delta_2 = A/2$.

For $l_{i_1} = l_{i_2}$ the following relationship holds:

$$r_{i_2} = |r_{i_1} + \Delta_1|_A = |r_{i_1} + A - \Delta_2|_A = |r_{i_1} - \Delta_2|_A\,.$$

Note the special case when $\Delta = 0$, in which case Type 1 is identical to Type 2 ; and also the special case when $\Delta_1 = \Delta_2 = A/2$, in which case:

$$r_{i_2} = |r_{i_1} + A/2|_A = |r_{i_1} - A/2|_A \,.$$

(1) — STANDARD REGULAR-NESTED CYLINDRICAL BRAIDS

$$\Delta = 0 \,; \qquad B^* \text{ and } P_{component} \text{ are coprime.}$$

$$P_{total} = P = \sum P_{component} = x + 2A - 2 \,,$$

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{A} \,,$$

$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \ldots \,.$$

Example:

Given $l_i = 3$; $r_i = 5$; $A = 5$.

Therefore:

$$\delta = |2(l_i + r_i) + \Delta|_A = |2(3 + 5) + 0|_5 = |16|_5 = 1 \,.$$

$l_i + r_i = 3 + 5 = 8 = Y + A$ thus $Y = 3$ and hence also $l_i + r_i = 3$.

$$x = cA + \delta = cA + 1 = 5c + 1 \,,$$

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{5} = c + 1 \qquad \text{for } l_i + r_i = 8 \,,$$

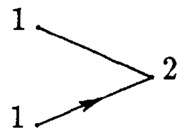
and

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{5} = c + 3 \qquad \text{for } l_i + r_i = 3 \,.$$

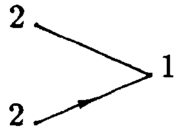
$$P = P_{total} = cA + 2A - 1 = 5c + 9 \,.$$

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

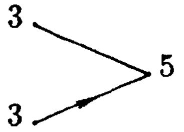
The first-return string-runs of the components are therefore as follows:



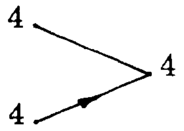
$P_{comp} = c + 3 \, , \qquad (x = 5c + 1) \, .$



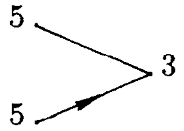
$P_{comp} = c + 3 \, , \qquad (x = 5c + 1) \, .$



$P_{comp} = c + 1 \, , \qquad (x = 5c + 1) \, .$



$P_{comp} = c + 1 \, , \qquad (x = 5c + 1) \, .$



$P_{comp} = c + 1 \, , \qquad (x = 5c + 1) \, .$

For $c = 2$ we obtain:

$$x = 5c + 1 = 11 \, .$$
$$P_{component} = 5 \quad \text{and} \quad P_{component} = 3 \, ,$$
$$P_{total} = 19 \, .$$

B^* **is coprime to 3 and 5 .**

For $c = 3$ we obtain:

$$x = 5c + 1 = 16 \, .$$
$$P_{component} = 6 \quad \text{and} \quad P_{component} = 4 \, ,$$
$$P_{total} = 24 \, .$$

B^* **is coprime to 2 and 3 .**

(2) — SEMI-STANDARD REGULAR-NESTED CYLINDRICAL BRAIDS

$\Delta = 0$; B^* and $P^*_{component}$ have a common divisor greater than 1.

$P^*_{component} = P_{component}$ as calculated in **(1)**.

All derivations are as in **(1)**.

See the Example in **(1)**:

For $c = 2$ we obtained:

$$x = 5c + 1 = 11\,.$$

$$P_{component} = 5 \quad \text{and} \quad P_{component} = 3\,.$$

Hence $$P^*_{component} = 5 \quad \text{and} \quad P^*_{component} = 3\,,$$

$$P_{total} = 19\,.$$

B^* has a divisor of 3 and/or 5.

For $c = 3$ we obtained:

$$x = 5c + 1 = 16\,.$$

$$P_{component} = 6 \quad \text{and} \quad P_{component} = 4\,.$$

Hence $$P^*_{component} = 6 \quad \text{and} \quad P^*_{component} = 4\,,$$

$$P_{total} = 24\,.$$

B^* has a divisor of 2 and/or 3.

(3) — PERFECT REGULAR-NESTED CYLINDRICAL BRAIDS

The Perfect Regular-Nested braids have only one component and hence are single-string braids.

Δ and A are coprime.

B^* and P_{total} are coprime.

Type 1: $\Delta_1 = \epsilon$ for $1 \leq \epsilon < A/2$,

Type 2: $\Delta_2 = A - \epsilon$ for $1 \leq \epsilon < A/2$.

Thus ϵ and A are coprime.

Since these braids are braided from one string and hence have only one component, we obtain:

$$P_{component} = P_{total} = x + 2A - 2\,.$$

$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \ldots\,.$$

Example 1:

Example 1(a) Type 1 Given $\epsilon = 1$; $l_i = 3$; $r_i = 4$; $A = 5$.

$$\Delta_1 = \epsilon = 1 \ : \qquad \delta = |2(3+4)+1|_5 = 5\,.$$

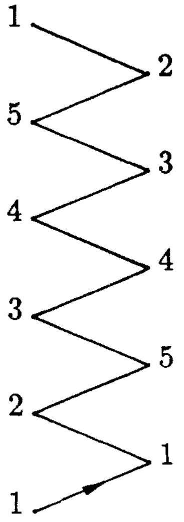
$l_i + r_i = 3 + 4 = 7$,

$Y + A = Y + 5 = 7$ therefore $Y = 2$ and hence also $l_i + r_i = 2$.

$$x = cA + \delta = 5c + 5\,,$$
$$P_{total} = P = 5c + 13\,.$$

	l_i	r_i		
	5	2	for	$l_i + r_i = 7$
	4	3	for	$l_i + r_i = 7$
given	**3**	**4**	**for**	$l_i + r_i = 7$
	2	5	for	$l_i + r_i = 7$
	1	1	for	$l_i + r_i = 2$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 5c + 5\,,$$
$$P = 5c + 13\,.$$

Example 1(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (1(a)).

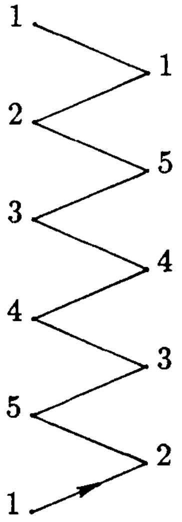
For $l_i = 3$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} + \Delta_1|_A = |4 + 1|_5 = 5\,.$$
$$l_i + r_i = 3 + 5 = 8\,,$$
$$Y + A = Y + 5 = 8 \qquad \text{therefore } Y = 3 \text{ and hence also } l_i + r_i = 3\,.$$
$$\Delta_2 = 5 - 1 = 4\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 5c + 5\,,$$
$$P = 5c + 13\,.$$

In this example for both types, we obtain the following:
For $c = 1$:

$$x = 5c + 5 = 10\,,$$
$$P = 5c + 13 = 18\,.$$

B^* should not have a divisor greater than 1 in common with 18 .

Therefore: **B^* is coprime with both 2 and 3 .**

For $c = 2$:

$$x = 5c + 5 = 15\,,$$
$$P = 5c + 13 = 23\,.$$

B^* should not have a divisor greater than 1 in common with 23 .

Therefore: **B^* is coprime with 23 .**

Example 2:

Example 2(a) Type 2 Given $\epsilon = 1$; $l_i = 3$; $r_i = 1$; $A = 5$.

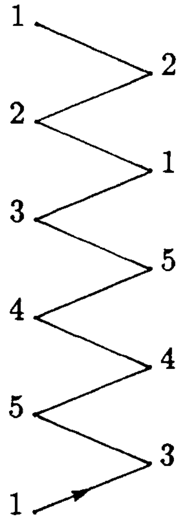
$$\Delta_1 = 1\,, \text{ and so } \Delta_2 = A - 1 = 4\,.$$
$$\delta = |2(3+1)+4|_5 = 2\,.$$
$$l_i + r_i = 3 + 1 = 4 = Y\,,$$

hence also $l_i + r_i = Y + A = 4 + 5 = 9\,.$

$$x = cA + \delta = 5c + 2\,,$$
$$P_{total} = P = 5c + 10\,.$$

	l_i	r_i		
	5	4	for	$l_i + r_i = 9$
	4	5	for	$l_i + r_i = 9$
given	**3**	**1**	**for**	$l_i + r_i = 4$
	2	2	for	$l_i + r_i = 4$
	1	3	for	$l_i + r_i = 4$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 5c + 2\,,$$
$$P = 5c + 10\,.$$

Example 2(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (2(a)).

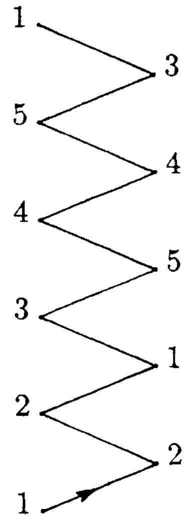
For $l_i = 3$, we obtain:

$$r_{i_2} = |r_{i_1} - \Delta_2|_A = |r_{i_1} - 4|_5 = 1\,,\ \text{hence} \qquad r_{i_1} = r_i = 5\,.$$
$$l_i + r_i = 3 + 5 = 8\,,$$
$$Y + A = Y + 5 = 8 \qquad \text{therefore } Y = 3 \text{ and hence also } l_i + r_i = 3\,.$$

As before, it is evident that $\delta_{type1} = \delta_{type2}$, and also that $c_{type1} = c_{type2}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 5c + 2\,,$$
$$P = 5c + 10\,.$$

In this example for both types, we obtain the following:
For $c = 2$:

$$x = 5c + 2 = 12\,,$$
$$P = 5c + 10 = 20\,.$$

B^* should not have a divisor greater than 1 in common with 20 .

Therefore: **B^* is coprime with both 2 and 5 .**

For $c = 3$:

$$x = 5c + 2 = 17\,,$$
$$P = 5c + 10 = 25\,.$$

B^* should not have a divisor greater than 1 in common with 25 .

Therefore: **B^* is coprime with 5 .**

Example 3 :

Example 3(a) Type 1 Given $\epsilon = 2$; $l_i = 2$; $r_i = 4$; $A = 5$.

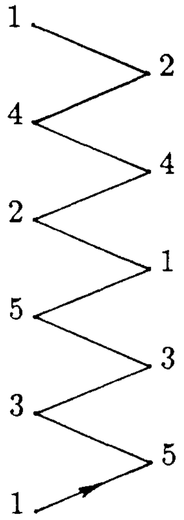
$$\epsilon = 2 \text{ implies } \Delta_1 = 2 : \qquad \delta = |2(2+4)+2|_5 = 4\,.$$
$$l_i + r_i = 2 + 4 = 6 = A + 1\,,$$

Hence for all $l_i + r_i$ we obtain the value 6.

$$x = cA + \delta = 5c + 4\,,$$
$$P_{total} = P = x + 2A - 2 = 5c + 12\,.$$

	l_i	r_i		
	5	1	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	1	5	for	$l_i + r_i = 6$

The first-return string-run is therefore as follows (with $\Delta = 2$):



$$x = 5c + 4\,,$$
$$P = 5c + 12\,.$$

Example 3(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (3(a)).

For $l_i = 2$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} + \Delta_1|_A = |4+2|_5 = 1\,.$$

$$l_i + r_i = 2 + 1 = 3\,,$$

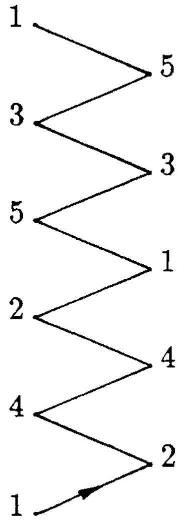
$$Y = 3 \qquad \text{therefore } Y + A = 8 \text{ and hence also } l_i + r_i = 8\,.$$

$$\Delta_2 = 5 - 2 = 3\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
given	**2**	**1**	**for**	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

The first-return string-run is therefore as follows (with $\Delta = 3$):



$$x = 5c + 4\,,$$
$$P = 5c + 12\,.$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 5c + 4 = 14\,,$$
$$P = 5c + 12 = 22\,.$$

Since B^* and P are coprime, B^* must be coprime with 2 and 11 .

Example 4:

Example 4(a) Type 2 Given $\epsilon = 2$; $l_i = 4$; $r_i = 1$; $A = 5$.

$$\epsilon = 2 \text{ implies } \Delta_2 = A - \epsilon = 3 \ : \qquad \delta = |2(4+1)+3|_5 = 3 .$$

$$l_i + r_i = 4 + 1 = 5 = Y ,$$

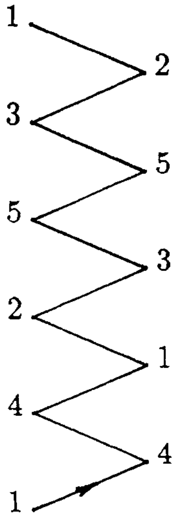
hence also $l_i + r_i = Y + A = 5 + 5 = 10$.

$$x = cA + \delta = 5c + 3 ,$$

$$P_{total} = P = 5c + 11 .$$

	l_i	r_i		
	5	5	for	$l_i + r_i = 10$
given	**4**	**1**	**for**	$l_i + r_i = 5$
	3	2	for	$l_i + r_i = 5$
	2	3	for	$l_i + r_i = 5$
	1	4	for	$l_i + r_i = 5$

The first-return string-run is therefore as follows (with $\Delta = 3$):



$$x = 5c + 3 ,$$
$$P = 5c + 11 .$$

Example 4(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (4(a)).

For $l_i = 4$, we obtain:

$$r_{i_2} = |r_{i_1} - \Delta_2|_A = |r_{i_1} - 3|_5 = 1 , \quad \text{hence} \qquad r_{i_1} = r_i = 4 .$$

$$\Delta_1 = \epsilon = A - \Delta_2 = 5 - 3 = 2 .$$

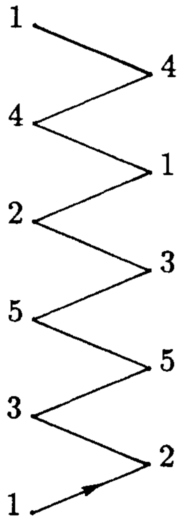
$$l_i + r_i = 4 + 4 = 8 ,$$

$Y + A = Y + 5 = 8$ therefore $Y = 3$ and hence also $l_i + r_i = 3$.

As before, it is evident that $\delta_{type1} = \delta_{type2}$, and also that $c_{type1} = c_{type2}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
given	**4**	**4**	**for**	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

The first-return string-run is therefore as follows (with $\Delta = 2$):



$$x = 5c + 3\,,$$
$$P = 5c + 11\,.$$

In this example for both Types, we obtain when $c = 1$ the following:

$$x = 5c + 3 = 8\,,$$
$$P = 5c + 11 = 16\,.$$

B^* should not have a divisor greater than 1 in common with 16,

Therefore: **B^* is coprime with 2 ; that is, B^* is odd.**

(4) — SEMI-PERFECT REGULAR-NESTED CYLINDRICAL BRAIDS

Δ and A are coprime.

B^* and P_{total} are **not** coprime, hence they have a common divisor greater than 1 .

Type 1: $\Delta_1 = \epsilon$ for $1 \le \epsilon < A/2$,
Type 2: $\Delta_2 = A - \epsilon$ for $1 \le \epsilon < A/2$.

Thus ϵ and A are coprime.

$$P_{total} = x + 2A - 2\,,$$
$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \ldots\,.$$

The Semi-Perfect Regular-Nested Cylindrical Braids are like the Perfect Regular-Nested Cylindrical Braids, except that they require more than one string in their construction.

All formulae derivations are as in (**3**) above.

Example 1:

Example 1(a) Type 1, and Example 1(b) Type 2

Using the same numerical values for the various parameters as in examples 1(a) and 1(b) above, under (**3**), we find that the only difference for the Semi-Perfect Regular-Nested Cylindrical Braids is that now:

For $c = 1$: B^* has a divisor of 2 and/or 3.

For $c = 2$: B^* has a divisor of 23.

Example 2:

Example 2(a) Type 2, and Example 2(b) Type 1

Using the same numerical values for the various parameters as in examples 2(a) and 2(b) above, under **(3)**, we find that the only difference for the Semi-Perfect Regular-Nested Cylindrical Braids is that now:

For $c = 2$: B^* has a divisor of 2 and/or 5.

For $c = 3$: B^* has a divisor of 5.

Example 3:

Example 3(a) Type 1, and Example 3(b) Type 2

Using the same numerical values for the various parameters as in examples 3(a) and 3(b) above, under **(3)**, we find that the only difference for the Semi-Perfect Regular-Nested Cylindrical Braids is that now B^* has a divisor of 2 and/or 11.

Example 4:

Example 4(a) Type 2, and Example 4(b) Type 1

Using the same numerical values for the various parameters as in examples 4(a) and 4(b) above, under **(3)**, we find that the only difference for the Semi-Perfect Regular-Nested Cylindrical Braids is that now B^* has a divisor of 2 and hence is even.

(5) — COMPOUND REGULAR-NESTED CYLINDRICAL BRAIDS

For braids in this sub-class, Δ can take any value in the range $2, \ldots, (A-2)$ which has with A a common divisor.

B^* and $P_{component}$ are coprime.

The number of passes is the same for each component of a given braid.

We define the number of passes per component by α. Hence the number of components of a given braid is A/α .

When a component is completed, α bight-boundaries on each edge of the braid have been visited, and then the equation $|1 + \alpha \cdot \Delta|_A = 1$ is satisfied. This implies that $\alpha \cdot \Delta$ is an integral multiple (say σ) of A ; that is $\alpha \cdot \Delta = \sigma \cdot A$; and so, finally, we can write:

$$\Delta = \frac{\sigma \cdot A}{\alpha} .$$

We may note that σ and α are coprime.

Type 1 : $\Delta_1 = \epsilon$ $\qquad 2 \leq \epsilon < A/2$,

Type 2 : $\Delta_2 = A - \epsilon$ $\qquad 2 \leq \epsilon < A/2$.

Note that Type 1 and Type 2 have the same Δ-value when $\Delta_1 = \Delta_2 = A/2$.

Furthermore, since Δ and A have a common divisor greater than 1 , ϵ and A are not coprime.

$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \dots ,$$
$$P_{component} = P_\alpha = \alpha(c + 4) + \left\{ \frac{\alpha \cdot \delta - 2(\Sigma l_i + \Sigma r_i)}{A} \right\} ,$$
$$P_{total} = \Sigma P_{component} = x + 2A - 2 .$$

Example 1 :

Given $l_i = 2$; $r_i = 1$; $A = 6$.

$A = 6$ has the divisors 2 and 3 ; thus $\alpha = 2$ or 3.

Let α be equal to 2 ; then $\Delta = \sigma A/2 = 3\sigma$.

But since $2 \le \Delta \le A - 2$, we obtain $\sigma = 1$ and hence $\Delta = 3$.

Example 1(a)

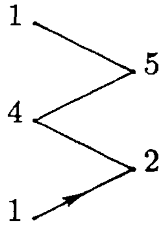
$$\delta = |2(2+1) + 3|_6 = 3 ,$$
$$x = cA + \delta = 6c + 3 ,$$
$$P = P_{total} = x + 2A - 2 = 6c + 13 .$$

$l_i + r_i = 2 + 1 = 3$,

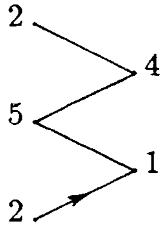
$Y = 3$ hence also $l_i + r_i = 3 + A = 3 + 6 = 9$.

	l_i	r_i		
	6	3	for	$l_i + r_i = 9$
	5	4	for	$l_i + r_i = 9$
	4	5	for	$l_i + r_i = 9$
	3	6	for	$l_i + r_i = 9$
given	**2**	**1**	**for**	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

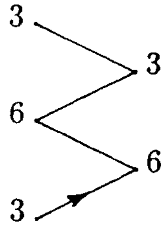
The first-return string-runs are therefore as follows (with $\Delta = 3$) :



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(1 + 4 + 2 + 5)}{6} = 2c + 5 .$$



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(2 + 5 + 1 + 4)}{6} = 2c + 5 .$$



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(3 + 6 + 6 + 3)}{6} = 2c + 3 .$$

Example 1(b)

We now give details of the mirror-image of the foregoing example (1(a)).

For $l_i = 2$, we obtain :

$$r_i = r_{i_2} = |r_{i_1} + \Delta_1|_A = |1+3|_6 = 4\,.$$

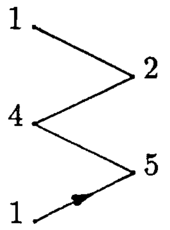
$$l_i + r_i = 2 + 4 = 6\,,$$

$$Y = 6 \qquad \text{hence also } l_i + r_i = 6 + A = 6 + 6 = 12\,.$$

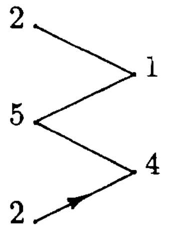
$$\Delta_2 = A - \Delta_1 = 6 - 3 = 3 = \Delta_1\,.$$

	l_i	r_i		
	6	6	for	$l_i + r_i = 12$
	5	1	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	1	5	for	$l_i + r_i = 6$

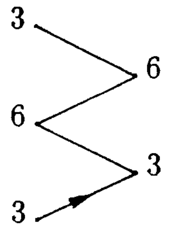
The first-return string-runs are therefore as follows (with $\Delta = 3$) :



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(1+4+5+2)}{6} = 2c + 5\,.$$



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(2+5+4+1)}{6} = 2c + 5\,.$$



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(3+6+3+6)}{6} = 2c + 3\,.$$

It is of course inconsequential which of the above two braids 1(a) or 1(b) is called Type 1 or Type 2. We shall however indicate the braids which have a half-cycle $_1\nearrow^1$; $_1\nearrow^2$ and $_1\nearrow^3$ as Type 1 braids and hence the braids which have a half-cycle $_1\nearrow^4$; $_1\nearrow^5$ and $_1\nearrow^6$ as Type 2 braids.

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 3 = 15\,,$$
$$P_{total} = P = 6c + 13 = 25\,,$$
$$P_{component} = 9 \text{ and } P_{component} = 7\,.$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 3 and 7.

Example 2:

Given $l_i = 2$; $r_i = 2$; $A = 6$.

Let α be equal to 3; then $\Delta = \dfrac{\sigma A}{3} = 2\sigma$.

Since $2 \le \Delta \le A - 2$, hence $2 \le \Delta \le 4$, the value of σ can be 1 or 2.

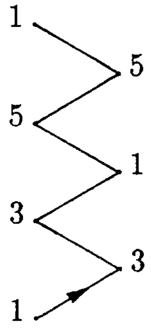
For $\sigma = 1$ we obtain $\Delta = 2$, and for $\sigma = 2$ we obtain $\Delta = 4$.

Example 2(a) Type 1

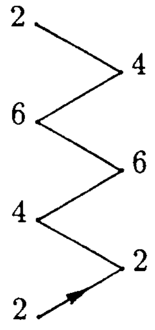
$$\Delta = 2\,.$$
$$\delta = |2(2+2)+2|_6 = 4\,,$$
$$x = cA + \delta = 6c + 4\,,$$
$$P = P_{total} = x + 2A - 2 = 6c + 14\,.$$
$$l_i + r_i = 2 + 2 = 4\,,$$
$$Y = 4 \qquad \text{hence also } l_i + r_i = 4 + A = 4 + 6 = 10\,.$$

	l_i	r_i		
	6	4	for	$l_i + r_i = 10$
	5	5	for	$l_i + r_i = 10$
	4	6	for	$l_i + r_i = 10$
	3	1	for	$l_i + r_i = 4$
given	**2**	**2**	**for**	$l_i + r_i = 4$
	1	3	for	$l_i + r_i = 4$

The first-return string-runs are therefore as follows (with $\Delta = 2$):



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(1+3+5+3+1+5)}{6} = 3c + 8\,\cdot$$



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(2+4+6+2+6+4)}{6} = 3c + 6\,\cdot$$

Example 2(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (2(a)).

For $l_i = 2$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} + \Delta_1|_A = |2+2|_6 = 4 .$$

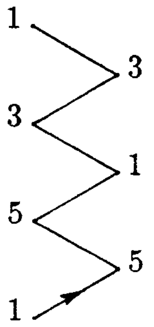
$$l_i + r_i = 2 + 4 = 6 ,$$

$$Y = 6 \qquad \text{hence also } l_i + r_i = 6 + A = 6 + 6 = 12 .$$

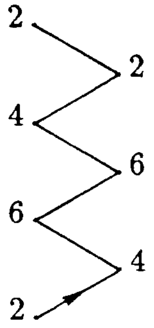
$$\Delta_2 = A - \Delta_1 = 6 - 2 = 4 .$$

	l_i	r_i		
	6	6	for	$l_i + r_i = 12$
	5	1	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	1	5	for	$l_i + r_i = 6$

The first-return string-runs are therefore as follows (with $\Delta = 4$):



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(1+5+3+5+1+3)}{6} = 3c + 8 \cdot$$



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(2+6+4+4+6+2)}{6} = 3c + 6 \cdot$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 4 = 16 ,$$

$$P_{total} = P = 6c + 14 = 26 ,$$

$$P_{component} = 14 \text{ and } P_{component} = 12 .$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 2 , 3 and 7 .

Example 3 :

Given $l_i = 2$; $r_i = 6$; $A = 6$.

Let α be equal to 3 ; then $\Delta = \frac{\sigma A}{3} = 2\sigma$.

Since $2 \leq \Delta \leq A - 2$, hence $2 \leq \Delta \leq 4$, the value of σ can be 1 or 2.

For $\sigma = 1$ we obtain $\Delta = 2$, and for $\sigma = 2$ we obtain $\Delta = 4$.

Example 3(a) Type 2

$$\Delta = 4 .$$

Note that we are dealing with a Type 2 since $\Delta = 4$ and hence $\epsilon = 2$ (Type 2: $\Delta_2 = A - \epsilon$ with $2 \leq \epsilon < A/2$) .

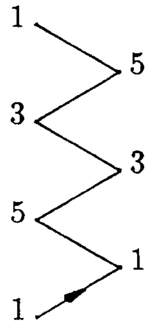
$$\delta = |2(2+6)+4|_6 = 2 ,$$
$$x = cA + \delta = 6c + 2 ,$$
$$P = P_{total} = x + 2A - 2 = 6c + 12 .$$

$l_i + r_i = 2 + 6 = 8$,

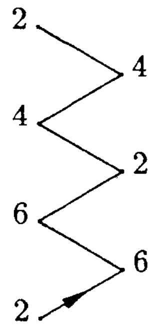
$Y + A = Y + 6 = 8$ therefore $Y = 2$ and hence also $l_i + r_i = 2$.

	l_i	r_i		
	6	2	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
given	**2**	**6**	**for**	$l_i + r_i = 8$
	1	1	for	$l_i + r_i = 2$

The first-return string-runs are therefore as follows (with $\Delta = 4$):



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(1+5+3+1+3+5)}{6} = 3c + 7 \cdot$$



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(2+6+4+6+2+4)}{6} = 3c + 5 \cdot$$

Example 3(b) Type 1

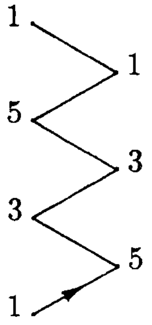
We now give details of the mirror-image of the foregoing example (3(a) Type 2).

For $l_i = 2$, we obtain:

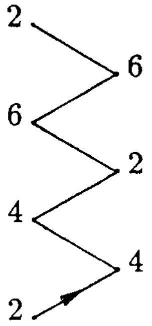
$$r_i = r_{i_1} = |r_{i_2} + \Delta_2|_A = |6+4|_6 = 4\,.$$
$$l_i + r_i = 2 + 4 = 6\,,$$
$$Y = 6 \qquad \text{hence also } l_i + r_i = 6 + A = 6 + 6 = 12\,.$$
$$\Delta_1 = A - \Delta_2 = 6 - 4 = 2\,.$$

	l_i	r_i		
	6	6	for	$l_i + r_i = 12$
	5	1	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	1	5	for	$l_i + r_i = 6$

The first-return string-runs are therefore as follows (with $\Delta = 2$):



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(1+3+5+5+3+1)}{6} = 3c + 7\,.$$



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(2+4+6+4+2+6)}{6} = 3c + 5\,.$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 2 = 14\,,$$
$$P_{total} = P = 6c + 12 = 24\,,$$
$$P_{component} = 13 \text{ and } P_{component} = 11\,.$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 13 and 11.

(6) — SEMI-COMPOUND REGULAR-NESTED CYLINDRICAL BRAIDS

For braids in this sub-class, Δ can take any value in the range $2, \ldots, (A-2)$ which has with A a common divisor.

B^* and $P^*_{component}$ have a common divisor greater than 1.

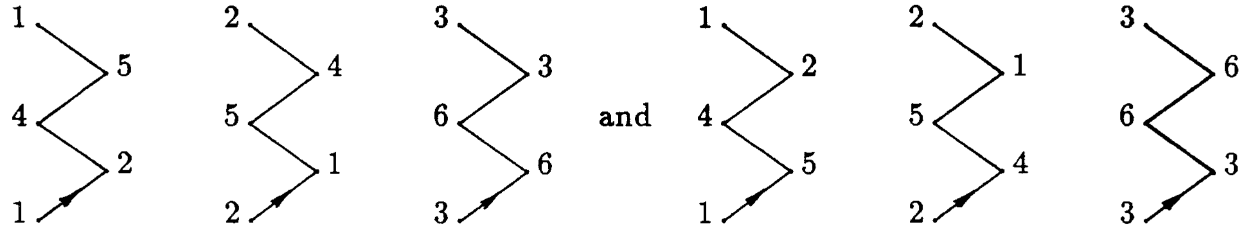
$P^*_{component} = P_{component}$ as calculated in **(5)**.

All derivations are as in **(5)**.

See the Examples in **(5)**:

Example 1:

Example 1(a) and 1(b)



$P^*_{comp} = 2c+5$ $P^*_{comp} = 2c+5$ $P^*_{comp} = 2c+3$ $P^*_{comp} = 2c+5$ $P^*_{comp} = 2c+5$ $P^*_{comp} = 2c+3$

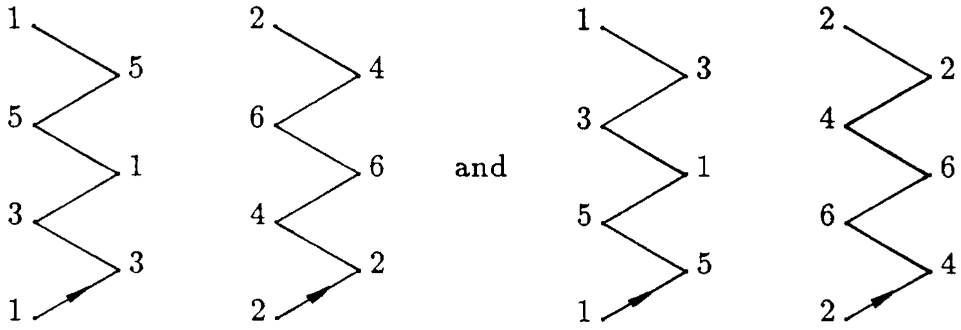
For $c = 2$ we obtain:

$$x = 6c + 3 = 15\,,$$
$$P_{total} = P = 6c + 13 = 25\,,$$
$$P^*_{component} = 9 \text{ and } P^*_{component} = 7\,.$$

B^* and $P^*_{component}$ have a common divisor of 3 and/or 7.

Example 2:

Example 2(a) and 2(b)



$P^*_{comp} = 3c+8$ $P^*_{comp} = 3c+6$ $P^*_{comp} = 3c+8$ $P^*_{comp} = 3c+6$

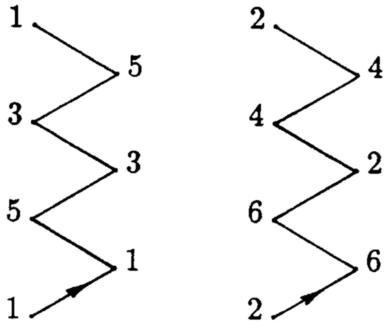
For $c = 2$ we obtain:

$$x = 6c + 4 = 16\,,$$
$$P_{total} = P = 6c + 14 = 26\,,$$
$$P^*_{component} = 14 \text{ and } P^*_{component} = 12\,.$$

B^* **and** $P^*_{component}$ **have a common divisor of 2 and/or 3 and/or 7.**

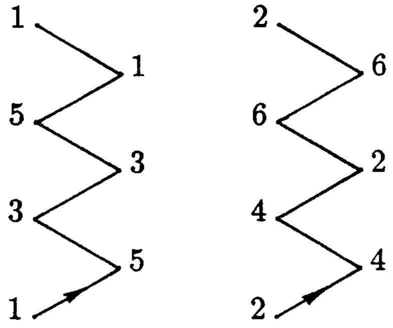
Example 3:

Example 3(a) and 3(b)



$P^*_{comp} = 3c + 7$ $P^*_{comp} = 3c + 5$

and



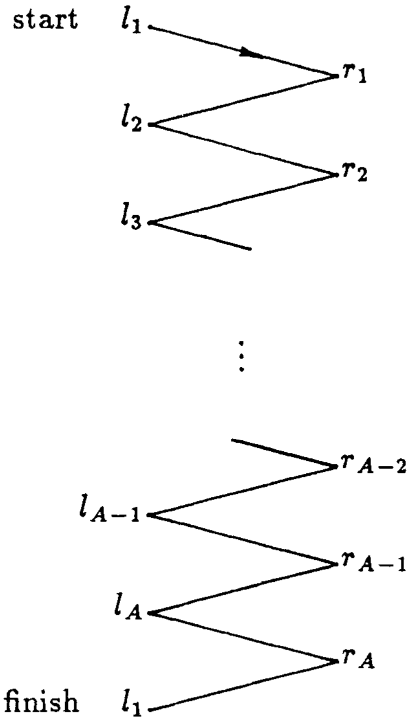
$P^*_{comp} = 3c + 7$ $P^*_{comp} = 3c + 5$

For $c = 2$ we obtain:

$$x = 6c + 2 = 14\,,$$
$$P_{total} = P = 6c + 12 = 24\,,$$
$$P^*_{component} = 13 \text{ and } P^*_{component} = 11\,.$$

B^* **and** $P^*_{component}$ **have a common divisor of 11 and/or 13.**

(II)
THE FORMULAE FOR BRAIDING DOWNWARDS



$$\begin{array}{ll}
l_2 = |l_1 - \Delta|_A & r_2 = |r_1 + \Delta|_A \\
l_3 = |l_2 - \Delta|_A & r_3 = |r_2 + \Delta|_A \\
l_4 = |l_3 - \Delta|_A & r_4 = |r_3 + \Delta|_A \\
\vdots & \vdots \\
l_{i+1} = |l_i - \Delta|_A & r_{i+1} = |r_i + \Delta|_A \\
\vdots & \vdots
\end{array}$$

In the above, $i = 1, 2, 3, \ldots, (A-1)$ and
Δ is an integer belonging to the set: $\{0, 1, 2, 3, \ldots, (A-2), (A-1)\}$.

All components of a given braid have the same Δ-value.

$\delta = |2(l_i + r_i) - \Delta|_A$ where δ is an integer belonging to the set: $\{1, 2, 3, \ldots, (A-1), A\}$.

All components of a given braid have the same δ-value.

For all half-cycles of a given braid, the value of $(l_i + r_i)$ is either equal to:
(1) $A + 1$
or
(2) Y, or $Y + A$, where $2 \leq Y \leq A$.

All components of a given braid have the same x-value, and since $x = cA + \delta$ it follows that all components of a given braid have the same c-value.

In general we have:

$$\begin{array}{lll}
\text{Type 1:} & \Delta_1 = \epsilon & \text{for } 1 \leq \epsilon < A/2\,, \\
\text{Type 2:} & \Delta_2 = A - \epsilon & \text{for } 1 \leq \epsilon < A/2\,. \\
\text{Thus} & \multicolumn{2}{l}{|l_{i_2} + r_{i_2}|_A = |l_{i_1} + r_{i_1} - \Delta_1|_A\,.}
\end{array}$$

Note that Type 1 and Type 2 have the same Δ-value when $\Delta_1 = \Delta_2 = A/2$.

For $l_{i_1} = l_{i_2}$ the following relationship holds:

$$r_{i_2} = |r_{i_1} - \Delta_1|_A = |r_{i_1} - A + \Delta_2|_A = |r_{i_1} + \Delta_2|_A\,.$$

Note the special case when $\Delta = 0$, in which case Type 1 is identical to Type 2; and also the special case when $\Delta_1 = \Delta_2 = A/2$, in which case:

$$r_{i_2} = |r_{i_1} - A/2|_A = |r_{i_1} + A/2|_A \,.$$

(1) — STANDARD REGULAR-NESTED CYLINDRICAL BRAIDS

$\Delta = 0$; B^* and $P_{component}$ are coprime.

$$P_{total} = P = \sum P_{component} = x + 2A - 2\,,$$

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{A}\,,$$

$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \ldots\,.$$

Example:

Given $l_i = 3$; $r_i = 5$; $A = 5$.

Therefore:

$$\delta = |2(l_i + r_i) - \Delta|_A = |2(3+5) - 0|_5 = |16|_5 = 1\,.$$

$l_i + r_i = 3 + 5 = 8 = Y + A$ thus $Y = 3$ and hence also $l_i + r_i = 3$.

$$x = cA + \delta = cA + 1 = 5c + 1\,,$$

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{5} = c + 1 \qquad \text{for} \quad l_i + r_i = 8\,,$$

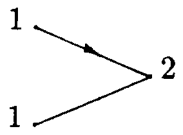
and

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{5} = c + 3 \qquad \text{for} \quad l_i + r_i = 3\,.$$

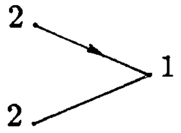
$$P = P_{total} = cA + 2A - 1 = 5c + 9\,.$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

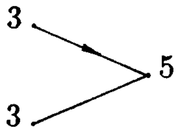
The first-return string-runs of the components are therefore as follows:



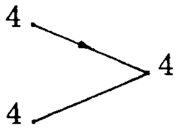
$P_{comp} = c + 3\,, \qquad (x = 5c + 1)\,.$



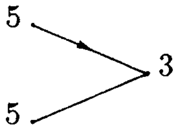
$P_{comp} = c + 3\,, \qquad (x = 5c + 1)\,.$



$P_{comp} = c + 1\,, \qquad (x = 5c + 1)\,.$



$P_{comp} = c + 1\,, \qquad (x = 5c + 1)\,.$



$P_{comp} = c + 1\,, \qquad (x = 5c + 1)\,.$

For $c = 2$ we obtain:

$$x = 5c + 1 = 11\,.$$
$$P_{component} = 5 \quad \text{and} \quad P_{component} = 3\,,$$
$$P_{total} = 19\,.$$

B^* is coprime to 3 and 5.

For $c = 3$ we obtain:

$$x = 5c + 1 = 16\,.$$
$$P_{component} = 6 \quad \text{and} \quad P_{component} = 4\,,$$
$$P_{total} = 24\,.$$

B^* is coprime to 2 and 3.

(2) — SEMI-STANDARD REGULAR-NESTED CYLINDRICAL BRAIDS

$\Delta = 0$; B^* and $P^*_{component}$ have a common divisor greater than 1.

$P^*_{component} = P_{component}$ as calculated in **(1)**.

All derivations are as in **(1)**.

See the Example in **(1)**:

For $c = 2$ we obtained:

$$x = 5c + 1 = 11 .$$

$$P_{component} = 5 \quad \text{and} \quad P_{component} = 3 .$$

Hence $$P^*_{component} = 5 \quad \text{and} \quad P^*_{component} = 3 ,$$

$$P_{total} = 19 .$$

B^* has a divisor of 3 and/or 5 .

For $c = 3$ we obtained:

$$x = 5c + 1 = 16 .$$

$$P_{component} = 6 \quad \text{and} \quad P_{component} = 4 .$$

Hence $$P^*_{component} = 6 \quad \text{and} \quad P^*_{component} = 4 ,$$

$$P_{total} = 24 .$$

B^* has a divisor of 2 and/or 3 .

(3) — PERFECT REGULAR-NESTED CYLINDRICAL BRAIDS

The Perfect Regular-Nested Cylindrical Braids have only one component and hence are single-string braids.

Δ and A are coprime.

B^* and P_{total} are coprime.

$$\text{Type 1}: \quad \Delta_1 = \epsilon \qquad \text{for} \quad 1 \leq \epsilon < A/2 ,$$

$$\text{Type 2}: \quad \Delta_2 = A - \epsilon \qquad \text{for} \quad 1 \leq \epsilon < A/2 .$$

Thus ϵ and A are coprime.

Since these braids are braided from one string and hence only have one component, we obtain:

$$P_{component} = P_{total} = x + 2A - 2 .$$

$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \ldots .$$

Example 1 :

Example 1(a) Type 1 Given $\epsilon = 1$; $l_i = 3$; $r_i = 5$; $A = 5$.

$$\Delta_1 = \epsilon = 1 \quad : \qquad \delta = |2(3+5) - 1|_5 = 5 .$$

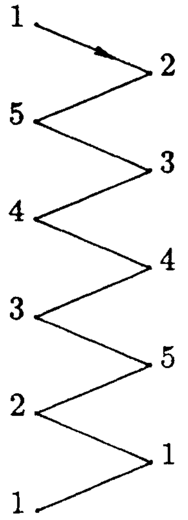
$$l_i + r_i = 3 + 5 = 8 ,$$

$$Y + A = Y + 5 = 8 \qquad \text{therefore } Y = 3 \text{ and hence also } l_i + r_i = 3 .$$

$$\begin{aligned} x &= cA + \delta = 5c + 5 \,, \\ P_{total} &= P = 5c + 13 \,. \end{aligned}$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$\begin{aligned} x &= 5c + 5 \,, \\ P &= 5c + 13 \,. \end{aligned}$$

Example 1(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (1(a)).

For $l_i = 3$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |5 - 1|_5 = 4 \,.$$

$$l_i + r_i = 3 + 4 = 7 \,,$$

$$Y + A = Y + 5 = 7 \qquad \text{therefore } Y = 2 \text{ and hence also } l_i + r_i = 2 \,.$$

$$\Delta_2 = 5 - 1 = 4 \,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	1	1	for	$l_i + r_i = 2$
	2	5	for	$l_i + r_i = 7$
given	**3**	**4**	**for**	$l_i + r_i = 7$
	4	3	for	$l_i + r_i = 7$
	5	2	for	$l_i + r_i = 7$

(2) — SEMI-STANDARD REGULAR-NESTED CYLINDRICAL BRAIDS

$\Delta = 0$; B^* and $P^*_{component}$ have a common divisor greater than 1.

$P^*_{component} = P_{component}$ as calculated in (**1**).

All derivations are as in (**1**).

See the Example in (**1**):

For $c = 2$ we obtained:

$$x = 5c + 1 = 11 .$$

$$P_{component} = 5 \quad \text{and} \quad P_{component} = 3 .$$

Hence $$P^*_{component} = 5 \quad \text{and} \quad P^*_{component} = 3 ,$$

$$P_{total} = 19 .$$

B^* has a divisor of 3 and/or 5 .

For $c = 3$ we obtained:

$$x = 5c + 1 = 16 .$$

$$P_{component} = 6 \quad \text{and} \quad P_{component} = 4 .$$

Hence $$P^*_{component} = 6 \quad \text{and} \quad P^*_{component} = 4 ,$$

$$P_{total} = 24 .$$

B^* has a divisor of 2 and/or 3 .

(3) — PERFECT REGULAR-NESTED CYLINDRICAL BRAIDS

The Perfect Regular-Nested Cylindrical Braids have only one component and hence are single-string braids.

Δ and A are coprime.

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Type 1 : $\Delta_1 = \epsilon$ for $1 \leq \epsilon < A/2$,

Type 2 : $\Delta_2 = A - \epsilon$ for $1 \leq \epsilon < A/2$.

Thus ϵ and A are coprime.

Since these braids are braided from one string and hence only have one component, we obtain:

$$P_{component} = P_{total} = x + 2A - 2 .$$

$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \ldots .$$

Example 1 :

Example 1(a) Type 1 Given $\epsilon = 1$; $l_i = 3$; $r_i = 5$; $A = 5$.

$$\Delta_1 = \epsilon = 1 \ : \qquad \delta = |2(3 + 5) - 1|_5 = 5 .$$

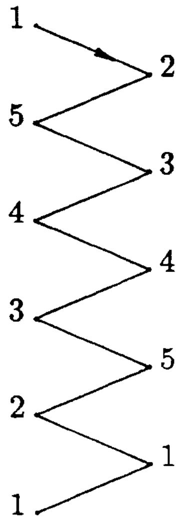
$l_i + r_i = 3 + 5 = 8$,

$Y + A = Y + 5 = 8$ therefore $Y = 3$ and hence also $l_i + r_i = 3$.

$$x = cA + \delta = 5c + 5\,,$$
$$P_{total} = P = 5c + 13\,.$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 5c + 5\,,$$
$$P = 5c + 13\,.$$

Example 1(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (1(a)).

For $l_i = 3$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |5 - 1|_5 = 4\,.$$

$$l_i + r_i = 3 + 4 = 7\,,$$

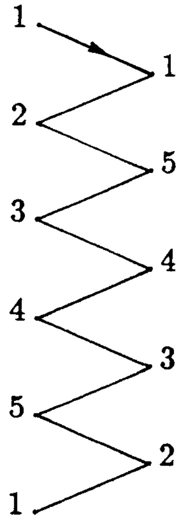
$$Y + A = Y + 5 = 7 \qquad \text{therefore } Y = 2 \text{ and hence also } l_i + r_i = 2\,.$$

$$\Delta_2 = 5 - 1 = 4\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	1	1	for	$l_i + r_i = 2$
	2	5	for	$l_i + r_i = 7$
given	**3**	**4**	**for**	$l_i + r_i = 7$
	4	3	for	$l_i + r_i = 7$
	5	2	for	$l_i + r_i = 7$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 5c + 5\,,$$
$$P = 5c + 13\,.$$

In this example for both types, we obtain the following:
For $c = 1$:

$$x = 5c + 5 = 10\,,$$
$$P = 5c + 13 = 18\,.$$

B^* should not have a divisor greater than 1 in common with 18 .

Therefore: **B^* is coprime with both 2 and 3 .**

For $c = 2$:

$$x = 5c + 5 = 15\,,$$
$$P = 5c + 13 = 23\,.$$

B^* should not have a divisor greater than 1 in common with 23 .

Therefore: **B^* is coprime with 23 .**

Example 2:

Example 2(a) Type 2 Given $\epsilon = 1$; $l_i = 3$; $r_i = 5$; $A = 5$.

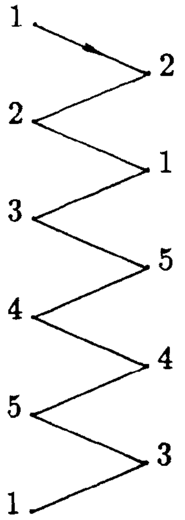
$$\Delta_1 = 1\,,\text{ and so } \Delta_2 = A - 1 = 4\,.$$
$$\delta = |2(3+5) - 4|_5 = 2\,.$$
$$l_i + r_i = 3 + 5 = 8 = Y + A\,,$$

hence also $l_i + r_i = Y = 3\,.$

$$x = cA + \delta = 5c + 2\,,$$
$$P_{total} = P = 5c + 10\,.$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 5c + 2\,,$$
$$P = 5c + 10\,.$$

Example 2(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (2(a)).

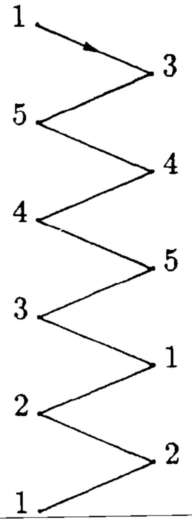
For $l_i = 3$, we obtain:

$$r_{i_2} = |r_{i_1} + \Delta_2|_A = |r_{i_1} + 4|_5 = 5\text{ , hence } \qquad r_{i_1} = r_i = 1\,.$$
$$l_i + r_i = 3 + 1 = 4 = Y\,,$$
$$Y + A = Y + 5 = 9 \qquad \text{hence also } l_i + r_i = 9\,.$$

As before, it is evident that $\delta_{type1} = \delta_{type2}$, and also that $c_{type1} = c_{type2}$ for mirror-imaged pairs.

	l_i	r_i		
	1	3	for	$l_i + r_i = 4$
	2	2	for	$l_i + r_i = 4$
given	**3**	**1**	**for**	$l_i + r_i = 4$
	4	5	for	$l_i + r_i = 9$
	5	4	for	$l_i + r_i = 9$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 5c + 2\,,$$
$$P = 5c + 10\,.$$

In this example for both types, we obtain the following:
For $c = 2$:

$$x = 5c + 2 = 12 ,$$
$$P = 5c + 10 = 20 .$$

B^* should not have a divisor greater than 1 in common with 20 .

Therefore: **B^* is coprime with both 2 and 5 .**

For $c = 3$:

$$x = 5c + 2 = 17 ,$$
$$P = 5c + 10 = 25 .$$

B^* should not have a divisor greater than 1 in common with 25 .

Therefore: **B^* is coprime with 5 .**

Example 3 :

Example 3(a) Type 1 Given $\epsilon = 2$; $l_i = 3$; $r_i = 5$; $A = 5$.

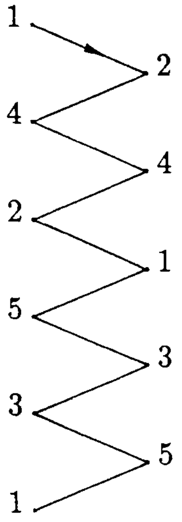
$$\epsilon = 2 \text{ implies } \Delta_1 = 2 : \qquad \delta = |2(3+5) - 2|_5 = 4 .$$
$$l_i + r_i = 3 + 5 = 8 = A + Y ,$$

hence also $l_i + r_i = Y = 3$.

$$x = cA + \delta = 5c + 4 ,$$
$$P_{total} = P = x + 2A - 2 = 5c + 12 .$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 2$):



$$x = 5c + 4 ,$$
$$P = 5c + 12 .$$

Example 3(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (3(a)).

For $l_i = 3$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |5-2|_5 = 3\,.$$

$$l_i + r_i = 3 + 3 = 6 = A + 1\,,$$

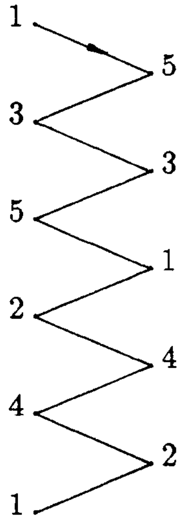
Hence for all $l_i + r_i$ we obtain the value 6.

$$\Delta_2 = 5 - 2 = 3\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	1	5	for	$l_i + r_i = 6$
	2	4	for	$l_i + r_i = 6$
given	**3**	**3**	**for**	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	5	1	for	$l_i + r_i = 6$

The first-return string-run is therefore as follows (with $\Delta = 3$):



$$x = 5c + 4\,,$$
$$P = 5c + 12\,.$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 5c + 4 = 14\,,$$
$$P = 5c + 12 = 22\,.$$

Since B^* and P are coprime, B^* must be coprime with 2 and 11.

Example 4:

Example 4(a) Type 2 Given $\epsilon = 2$; $l_i = 3$; $r_i = 5$; $A = 5$.

$$\epsilon = 2 \text{ implies } \Delta_2 = A - \epsilon = 3 \ : \qquad \delta = |2(3+5) - 3|_5 = 3 .$$

$$l_i + r_i = 3 + 5 = 8 = Y + A ,$$

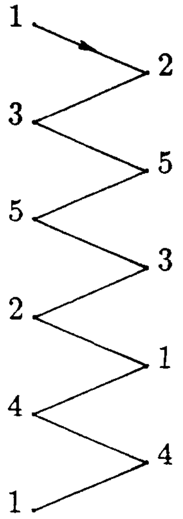
hence also $l_i + r_i = Y = 3$.

$$x = cA + \delta = 5c + 3 ,$$

$$P_{total} = P = 5c + 11 .$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 3$):



$$x = 5c + 3 ,$$
$$P = 5c + 11 .$$

Example 4(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (4(a)).

For $l_i = 3$, we obtain:

$$r_{i_2} = |r_{i_1} + \Delta_2|_A = |r_{i_1} + 3|_5 = 5 , \quad \text{hence} \qquad r_{i_1} = r_i = 2 .$$

$$\Delta_1 = \epsilon = A - \Delta_2 = 5 - 3 = 2 .$$

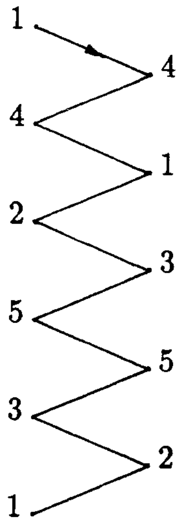
$$l_i + r_i = 3 + 2 = 5 = Y ,$$

$$Y + A = Y + 5 = 10 \qquad \text{hence also } l_i + r_i = 10 .$$

As before, it is evident that $\delta_{type1} = \delta_{type2}$, and also that $c_{type1} = c_{type2}$ for mirror-imaged pairs.

	l_i	r_i		
	1	4	for	$l_i + r_i = 5$
	2	3	for	$l_i + r_i = 5$
given	**3**	**2**	**for**	$l_i + r_i = 5$
	4	1	for	$l_i + r_i = 5$
	5	5	for	$l_i + r_i = 10$

The first-return string-run is therefore as follows (with $\Delta = 2$):



$$x = 5c + 3\,,$$
$$P = 5c + 11\,.$$

In this example for both Types, we obtain when $c = 1$ the following:

$$x = 5c + 3 = 8\,,$$
$$P = 5c + 11 = 16\,.$$

B^* should not have a divisor greater than 1 in common with 16 ,

Therefore: **B^* is coprime with 2 ; that is, B^* is odd .**

(4) — SEMI-PERFECT REGULAR-NESTED CYLINDRICAL BRAIDS

Δ and A are coprime.

B^* and P_{total} are not coprime, hence they have a common divisor greater than 1 .

Type 1: $\Delta_1 = \epsilon$ for $1 \leq \epsilon < A/2\,,$

Type 2: $\Delta_2 = A - \epsilon$ for $1 \leq \epsilon < A/2\,.$

Thus ϵ and A are coprime.

$$P_{total} = x + 2A - 2\,,$$
$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \ldots\,.$$

The Semi-Perfect Regular-Nested Cylindrical Braids are like the Perfect Regular-Nested Cylindrical Braids, except that they require more than one string in their construction.

All formulae derivations are as in (**3**) above.

Example 1 :

Example 1(a) Type 1, and Example 1(b) Type 2

Using the same numerical values for the various parameters as in examples 1(a) and 1(b) above, under (**3**), we find that the only difference for the Semi-Perfect Regular-Nested Cylindrical Braids is that now:

For $c = 1$: B^* has a divisor of 2 and/or 3.

For $c = 2$: B^* has a divisor of 23.

Example 2:

Example 2(a) Type 2, and Example 2(b) Type 1

Using the same numerical values for the various parameters as in examples 2(a) and 2(b) above, under (**3**), we find that the only difference for the Semi-Perfect Regular-Nested Cylindrical Braids is that now:

For $c = 2$: B^* has a divisor of 2 and/or 5.

For $c = 3$: B^* has a divisor of 5.

Example 3:

Example 3(a) Type 1, and Example 3(b) Type 2

Using the same numerical values for the various parameters as in examples 3(a) and 3(b) above, under (**3**), we find that the only difference for the Semi-Perfect Regular-Nested Cylindrical Braids is that now B^* has a divisor of 2 and/or 11.

Example 4:

Example 4(a) Type 2, and Example 4(b) Type 1

Using the same numerical values for the various parameters as in examples 4(a) and 4(b) above, under (**3**), we find that the only difference for the Semi-Perfect Regular-Nested Cylindrical Braids is that now B^* has a divisor of 2 and hence is even.

(5) — COMPOUND REGULAR-NESTED CYLINDRICAL BRAIDS

For braids in this sub-class, Δ can take any value in the range $2, \ldots, (A-2)$ which has with A a common divisor.

B^* and $P_{component}$ are coprime.

The number of passes is the same for each component of a given braid.

We define the number of passes per component by α. Hence the number of components of a given braid is A/α.

When a component is completed, α bight-boundaries on each edge of the braid have been visited, and then the equation $|1 + \alpha \cdot \Delta|_A = 1$ is satisfied. This implies that $\alpha \cdot \Delta$ is an integral multiple (say σ) of A; that is $\alpha \cdot \Delta = \sigma \cdot A$; and so, finally, we can write:

$$\Delta = \frac{\sigma \cdot A}{\alpha} .$$

We may note that σ and α are coprime.

Type 1 : $\Delta_1 = \epsilon$ $\quad 2 \leq \epsilon < A/2$,

Type 2 : $\Delta_2 = A - \epsilon$ $\quad 2 \leq \epsilon < A/2$.

Note that Type 1 and Type 2 have the same Δ-value when $\Delta_1 = \Delta_2 = A/2$.

Furthermore, since Δ and A have a common divisor greater than 1, ϵ and A are not coprime.

$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \dots ,$$

$$P_{component} = P_{\alpha} = \alpha(c+4) + \left\{ \frac{\alpha \cdot \delta - 2(\Sigma l_i + \Sigma r_i)}{A} \right\} ,$$

$$P_{total} = \Sigma P_{component} = x + 2A - 2 .$$

Example 1:

Given $l_i = 2$; $r_i = 4$; $A = 6$.

$A = 6$ has the divisors 2 and 3; thus $\alpha = 2$ or 3.

Let α be equal to 2; then $\Delta = \sigma A/2 = 3\sigma$.

But since $2 \leq \Delta \leq A - 2$, we obtain $\sigma = 1$ and hence $\Delta = 3$.

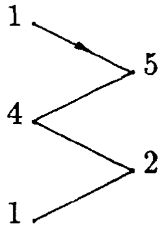
Example 1(a)

$$\delta = |2(2+4) - 3|_6 = 3 ,$$
$$x = cA + \delta = 6c + 3 ,$$
$$P = P_{total} = x + 2A - 2 = 6c + 13 .$$

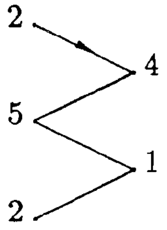
$$l_i + r_i = 2 + 4 = 6 ,$$
$$Y = 6 \qquad \text{hence also} \quad l_i + r_i = Y + A = 6 + 6 = 12 .$$

	l_i	r_i		
	1	5	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	5	1	for	$l_i + r_i = 6$
	6	6	for	$l_i + r_i = 12$

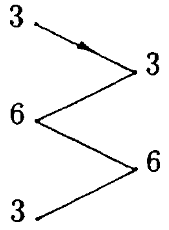
The first-return string-runs are therefore as follows (with $\Delta = 3$):



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(1+4+5+2)}{6} = 2c + 5 \cdot$$



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(2+5+4+1)}{6} = 2c + 5 \cdot$$



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(3+6+3+6)}{6} = 2c + 3 \cdot$$

Example 1(b)

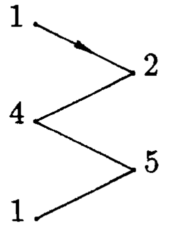
We now give details of the mirror-image of the foregoing example (1(a)).

For $l_i = 2$, we obtain:

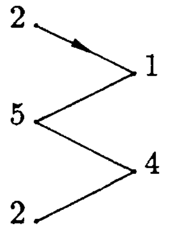
$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |4-3|_6 = 1\,.$$
$$l_i + r_i = 2 + 1 = 3\,,$$
$$Y = 3 \qquad \text{hence also } l_i + r_i = Y + A = 3 + 6 = 9\,.$$
$$\Delta_2 = A - \Delta_1 = 6 - 3 = 3 = \Delta_1\,.$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
given	**2**	**1**	**for**	$l_i + r_i = 3$
	3	6	for	$l_i + r_i = 9$
	4	5	for	$l_i + r_i = 9$
	5	4	for	$l_i + r_i = 9$
	6	3	for	$l_i + r_i = 9$

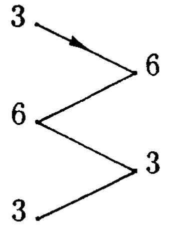
The first-return string-runs are therefore as follows (with $\Delta = 3$):



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(1+4+2+5)}{6} = 2c + 5\ .$$



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(2+5+1+4)}{6} = 2c + 5\ .$$



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(3+6+6+3)}{6} = 2c + 3\ .$$

It is of course inconsequential which of the above two braids 1(a) or 1(b) is called Type 1 or Type 2. We shall however indicate the braids which have a half-cycle ${}^{1}\searrow_{1}$; ${}^{1}\searrow_{2}$ and ${}^{1}\searrow_{3}$ as Type 1 braids and hence the braids which have a half-cycle ${}^{1}\searrow_{4}$; ${}^{1}\searrow_{5}$ and ${}^{1}\searrow_{6}$ as Type 2 braids.

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 3 = 15\,,$$
$$P_{total} = P = 6c + 13 = 25\,,$$
$$P_{component} = 9 \text{ and } P_{component} = 7\,.$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 3 and 7.

Example 2:

Given $l_i = 2$; $r_i = 4$; $A = 6$.

Let α be equal to 3; then $\Delta = \dfrac{\sigma A}{3} = 2\sigma$.

Since $2 \leq \Delta \leq A - 2$, hence $2 \leq \Delta \leq 4$, the value of σ can be 1 or 2.

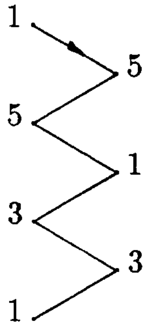
For $\sigma = 1$ we obtain $\Delta = 2$, and for $\sigma = 2$ we obtain $\Delta = 4$.

Example 2(a) Type 1

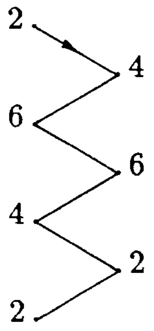
$$\Delta = 2\,.$$
$$\delta = |2(2+4) - 2|_6 = 4\,,$$
$$x = cA + \delta = 6c + 4\,,$$
$$P = P_{total} = x + 2A - 2 = 6c + 14\,.$$
$$l_i + r_i = 2 + 4 = 6\,,$$
$$Y = 6 \qquad \text{hence also } l_i + r_i = Y + A = 6 + 6 = 12\,.$$

	l_i	r_i		
	1	5	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	5	1	for	$l_i + r_i = 6$
	6	6	for	$l_i + r_i = 12$

The first-return string-runs are therefore as follows (with $\Delta = 2$):



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(1+5+3+5+1+3)}{6} = 3c + 8\,.$$



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(2+6+4+4+6+2)}{6} = 3c + 6\,.$$

Example 2(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (2(a)).

For $l_i = 2$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |4-2|_6 = 2\,.$$

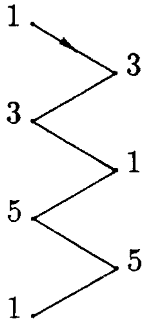
$$l_i + r_i = 2 + 2 = 4\,,$$

$$Y = 4 \qquad \text{hence also } l_i + r_i = Y + A = 4 + 6 = 10\,.$$

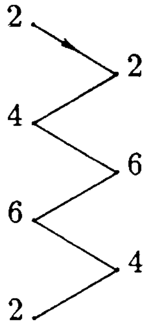
$$\Delta_2 = A - \Delta_1 = 6 - 2 = 4\,.$$

	l_i	r_i		
	1	3	for	$l_i + r_i = 4$
given	**2**	**2**	**for**	$l_i + r_i = 4$
	3	1	for	$l_i + r_i = 4$
	4	6	for	$l_i + r_i = 10$
	5	5	for	$l_i + r_i = 10$
	6	4	for	$l_i + r_i = 10$

The first-return string-runs are therefore as follows (with $\Delta = 4$):



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(1+3+5+3+1+5)}{6} = 3c + 8\,\cdot$$



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(2+4+6+2+6+4)}{6} = 3c + 6\,\cdot$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 4 = 16\,,$$

$$P_{total} = P = 6c + 14 = 26\,,$$

$$P_{component} = 14 \text{ and } P_{component} = 12\,.$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 2, 3 and 7.

Example 3 :

Given $l_i = 2$; $r_i = 4$; $A = 6$.

Let α be equal to 3 ; then $\Delta = \dfrac{\sigma A}{3} = 2\sigma$.

Since $2 \leq \Delta \leq A - 2$, hence $2 \leq \Delta \leq 4$, the value of σ can be 1 or 2.

For $\sigma = 1$ we obtain $\Delta = 2$, and for $\sigma = 2$ we obtain $\Delta = 4$.

Example 3(a) Type 2

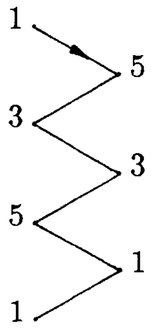
$$\Delta = 4 .$$

Note that we are dealing with a Type 2 since $\Delta = 4$ and hence $\epsilon = 2$ (Type 2: $\Delta_2 = A - \epsilon$ with $2 \leq \epsilon < A/2$) .

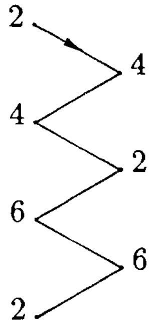
$$\delta = |2(2+4) - 4|_6 = 2 ,$$
$$x = cA + \delta = 6c + 2 ,$$
$$P = P_{total} = x + 2A - 2 = 6c + 12 .$$
$$l_i + r_i = 2 + 4 = 6 ,$$
$$Y = 6 \qquad \text{hence also } l_i + r_i = Y + A = 6 + 6 = 12 .$$

	l_i	r_i		
	1	5	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	5	1	for	$l_i + r_i = 6$
	6	6	for	$l_i + r_i = 12$

The first-return string-runs are therefore as follows (with $\Delta = 4$) :



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(1+3+5+5+3+1)}{6} = 3c + 7 \cdot$$



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(2+4+6+4+2+6)}{6} = 3c + 5 \cdot$$

Example 3(b) Type 1

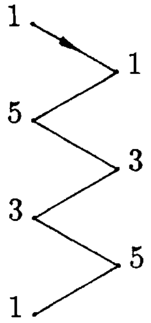
We now give details of the mirror-image of the foregoing example (3(a) Type 2).

For $l_i = 2$, we obtain:

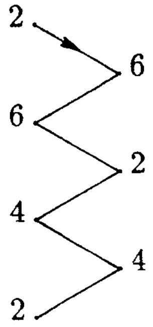
$$r_i = r_{i_1} = |r_{i_2} - \Delta_2|_A = |4-4|_6 = 6 \,.$$
$$l_i + r_i = 2 + 6 = 8 = Y + A \,,$$
$$Y = 2 \qquad \text{hence also } l_i + r_i = Y = 2 \,.$$
$$\Delta_1 = A - \Delta_2 = 6 - 4 = 2 \,.$$

	l_i	r_i		
	1	1	for	$l_i + r_i = 2$
given	**2**	**6**	**for**	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$
	6	2	for	$l_i + r_i = 8$

The first-return string-runs are therefore as follows (with $\Delta = 2$):



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(1+5+3+1+3+5)}{6} = 3c + 7 \cdot$$



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(2+6+4+6+2+4)}{6} = 3c + 5 \cdot$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 2 = 14 \,,$$
$$P_{total} = P = 6c + 12 = 24 \,,$$
$$P_{component} = 13 \text{ and } P_{component} = 11 \,.$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 13 and 11 .

(6) — SEMI-COMPOUND REGULAR-NESTED CYLINDRICAL BRAIDS

For braids in this sub-class, Δ can take any value in the range $2, \ldots, (A-2)$ which has with A a common divisor.

B^* and $P^*_{component}$ have a common divisor greater than 1.

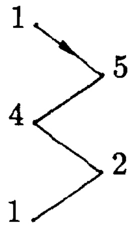
$P^*_{component} = P_{component}$ as calculated in (**5**).

All derivations are as in (**5**).

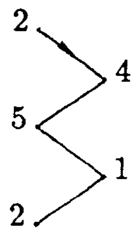
See the Examples in (**5**):

Example 1:

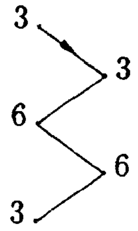
Example 1(a) and 1(b)



$P^*_{comp} = 2c+5$

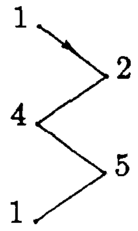


$P^*_{comp} = 2c+5$

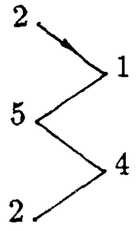


$P^*_{comp} = 2c+3$

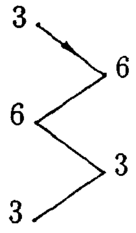
and



$P^*_{comp} = 2c+5$



$P^*_{comp} = 2c+5$



$P^*_{comp} = 2c+3$

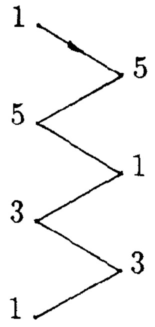
For $c = 2$ we obtain:

$$x = 6c + 3 = 15\,,$$
$$P_{total} = P = 6c + 13 = 25\,,$$
$$P^*_{component} = 9 \text{ and } P^*_{component} = 7\,.$$

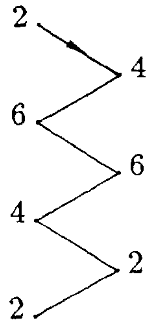
B^* and $P^*_{component}$ **have a common divisor of 3 and/or 7.**

Example 2:

Example 2(a) and 2(b)

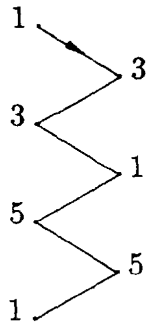


$P^*_{comp} = 3c+8$

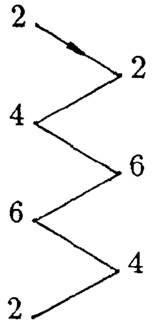


$P^*_{comp} = 3c+6$

and



$P^*_{comp} = 3c+8$



$P^*_{comp} = 3c+6$

Example 3(b) Type 1

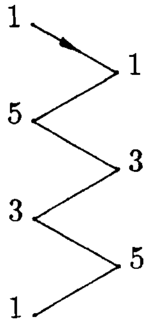
We now give details of the mirror-image of the foregoing example (3(a) Type 2).

For $l_i = 2$, we obtain:

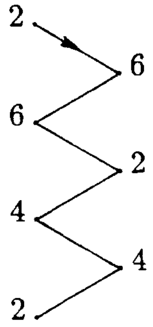
$$\begin{aligned} r_i &= r_{i_1} = |r_{i_2} - \Delta_2|_A = |4-4|_6 = 6\,. \\ l_i + r_i &= 2 + 6 = 8 = Y + A\,, \\ Y &= 2 \qquad \text{hence also } l_i + r_i = Y = 2\,. \\ \Delta_1 &= A - \Delta_2 = 6 - 4 = 2\,. \end{aligned}$$

	l_i	r_i		
	1	1	for	$l_i + r_i = 2$
given	**2**	**6**	**for**	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$
	6	2	for	$l_i + r_i = 8$

The first-return string-runs are therefore as follows (with $\Delta = 2$):



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(1+5+3+1+3+5)}{6} = 3c + 7\,.$$



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(2+6+4+6+2+4)}{6} = 3c + 5\,.$$

In this example for both Types, we obtain when $c = 2$ the following:

$$\begin{aligned} x &= 6c + 2 = 14\,, \\ P_{total} &= P = 6c + 12 = 24\,, \\ P_{component} &= 13 \text{ and } P_{component} = 11\,. \end{aligned}$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 13 and 11.

(6) — SEMI-COMPOUND REGULAR-NESTED CYLINDRICAL BRAIDS

For braids in this sub-class, Δ can take any value in the range $2, \ldots, (A-2)$ which has with A a common divisor.

$$B^* \text{ and } P^*_{component} \text{ have a common divisor greater than } 1.$$

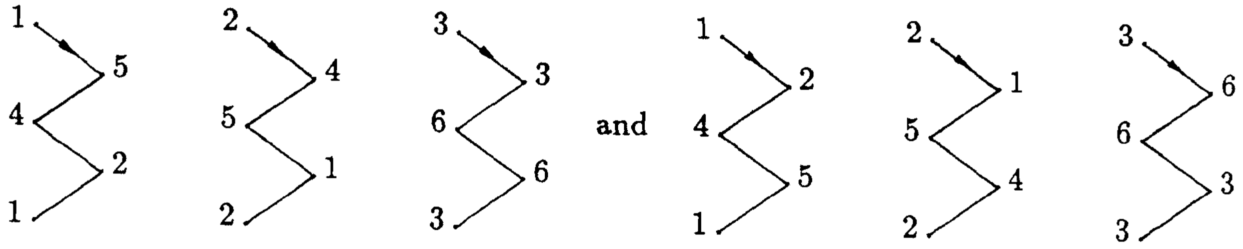
$$P^*_{component} = P_{component} \text{ as calculated in } (\mathbf{5}).$$

All derivations are as in (**5**).

See the Examples in (**5**):

Example 1 :

Example 1(a) and 1(b)



$P^*_{comp} = 2c+5$ $\quad$ $P^*_{comp} = 2c+5$ $\quad$ $P^*_{comp} = 2c+3$ $\quad$ $P^*_{comp} = 2c+5$ $\quad$ $P^*_{comp} = 2c+5$ $\quad$ $P^*_{comp} = 2c+3$

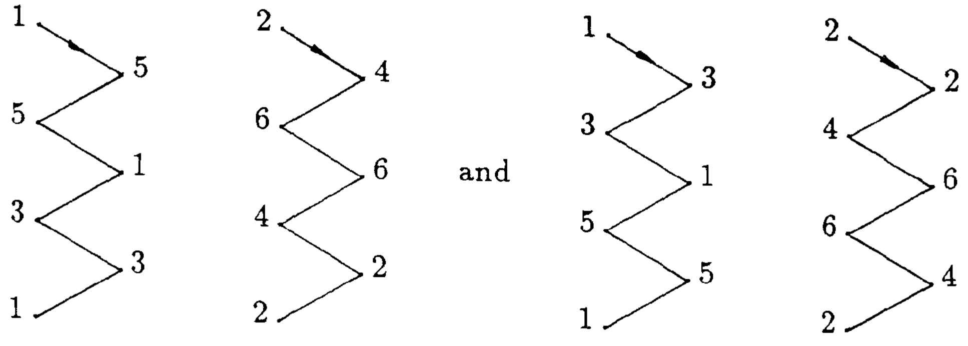
For $c = 2$ we obtain:

$$x = 6c + 3 = 15\,,$$
$$P_{total} = P = 6c + 13 = 25\,,$$
$$P^*_{component} = 9 \text{ and } P^*_{component} = 7\,.$$

B^* and $P^*_{component}$ **have a common divisor of 3 and/or 7 .**

Example 2 :

Example 2(a) and 2(b)



$P^*_{comp} = 3c+8$ $\quad$ $P^*_{comp} = 3c+6$ $\quad$ $P^*_{comp} = 3c+8$ $\quad$ $P^*_{comp} = 3c+6$

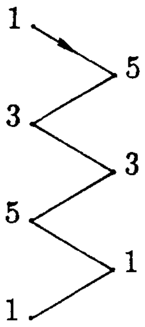
For $c = 2$ we obtain :

$$\begin{aligned} x &= 6c + 4 = 16\,, \\ P_{total} &= P = 6c + 14 = 26\,, \\ P^*_{component} &= 14 \text{ and } P^*_{component} = 12\,. \end{aligned}$$

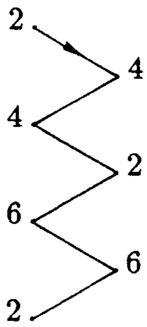
B^* and $P^*_{component}$ have a common divisor of 2 and/or 3 and/or 7 .

Example 3 :

Example 3(a) and 3(b)

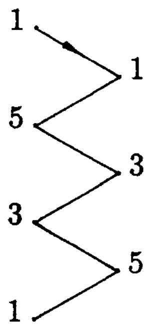


$P^*_{comp} =$ $3c + 7$

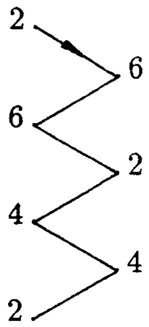


$P^*_{comp} =$ $3c + 5$

and



$P^*_{comp} =$ $3c + 7$



$P^*_{comp} =$ $3c + 5$

For $c = 2$ we obtain :

$$\begin{aligned} x &= 6c + 2 = 14\,, \\ P_{total} &= P = 6c + 12 = 24\,, \\ P^*_{component} &= 13 \text{ and } P^*_{component} = 11\,. \end{aligned}$$

B^* and $P^*_{component}$ have a common divisor of 11 and/or 13 .

Appendix 2

THE HERRINGBONE PINEAPPLE KNOT SUB-CLASS

This Appendix is a summary of the properties of the four Herringbone Pineapple Knot sub-families and the six Broken-Herringbone Pineapple Knot sub-families.

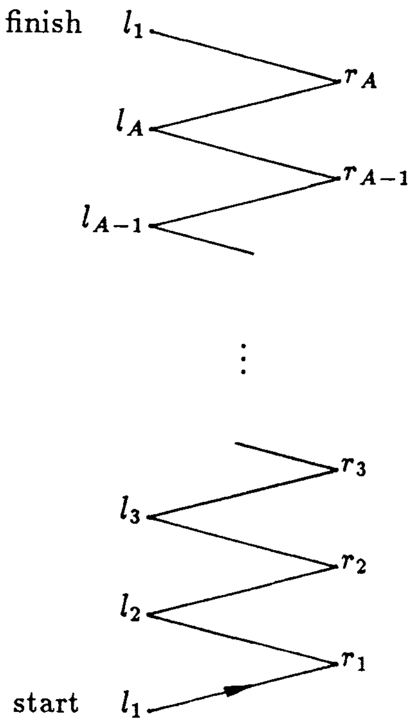
- The Herringbone Pineapple Knot sub-class
  - Herringbone Pineapple Knots
    - 1) Standard Herringbone Pineapple Knots.
    - 2) Semi-Standard Herringbone Pineapple Knots.
    - 3) Perfect Herringbone Pineapple Knots.
    - 4) Semi-Perfect Herringbone Pineapple Knots.
  - Broken-Herringbone Pineapple Knots
    - 5) Standard Broken-Herringbone Pineapple Knots.
    - 6) Semi-Standard Broken-Herringbone Pineapple Knots.
    - 7) Perfect Broken-Herringbone Pineapple Knots.
    - 8) Semi-Perfect Broken-Herringbone Pineapple Knots.
    - 9) Compound Broken-Herringbone Pineapple Knots.
    - 10) Semi-Compound Broken-Herringbone Pineapple Knots.

Knots can be braided in two ways, according as their cycles are laid down from lower-left to upper-left (upwards), or from upper-left to lower-left (downwards). The formulae which express their string-run characteristics depend on which of these ways is used. Therefore we give the formulae in two Sections, I and II. Section I deals with the case in which the braiding is upwards; whereas Section II treats the case where the braiding is downwards.

The examples used in Section II match the corresponding ones in Section I

(I)

THE FORMULAE FOR BRAIDING UPWARDS



$$\begin{array}{rlcrl}
l_2 &= |l_1+\Delta|_A & & r_2 &= |r_1-\Delta|_A \\
l_3 &= |l_2+\Delta|_A & & r_3 &= |r_2-\Delta|_A \\
l_4 &= |l_3+\Delta|_A & & r_4 &= |r_3-\Delta|_A \\
&\vdots & & &\vdots \\
l_{i+1} &= |l_i+\Delta|_A & & r_{i+1} &= |r_i-\Delta|_A \\
&\vdots & & &\vdots
\end{array}$$

In the above, $i = 1, 2, 3, \ldots, (A-1)$ and
Δ is an integer belonging to the set: $\{0, 1, 2, 3, \ldots, (A-2), (A-1)\}$.

All components of a given knot have the same Δ-value.

$\delta = |2(l_i + r_i) + \Delta|_A$ where δ is an integer belonging to the set: $\{1, 2, 3, \ldots, (A-1), A\}$.

All components of a given knot have the same δ-value.

For all half-cycles of a given knot, the value of $(l_i + r_i)$ is either equal to:
(1) $A+1$
or
(2) Y, or $Y + A$, where $2 \leq Y \leq A$.

All components of a given knot have the same x-value, and since $x = cA+\delta$ it follows that all components of a given knot have the same c-value.

In general we have:

$$\begin{array}{lll}
\text{Type 1:} & \Delta_1 = \epsilon & \text{for } 1 \leq \epsilon < A/2\,, \\
\text{Type 2:} & \Delta_2 = A - \epsilon & \text{for } 1 \leq \epsilon < A/2\,.
\end{array}$$

$$\text{Thus} \quad |l_{i_2} + r_{i_2}|_A = |l_{i_1} + r_{i_1} + \Delta_1|_A\,.$$

Note that Type 1 and Type 2 have the same Δ-value when $\Delta_1 = \Delta_2 = A/2$.

For $l_{i_1} = l_{i_2}$ the following relationship holds:

$$r_{i_2} = |r_{i_1} + \Delta_1|_A = |r_{i_1} + A - \Delta_2|_A = |r_{i_1} - \Delta_2|_A\,.$$

Note the special case when $\Delta = 0$, in which case Type 1 is identical to Type 2; and also the special case when $\Delta_1 = \Delta_2 = A/2$, in which case:

$$r_{i_2} = |r_{i_1} + A/2|_A = |r_{i_1} - A/2|_A .$$

(1) — STANDARD HERRINGBONE PINEAPPLE KNOTS

$$\Delta = 0 ; \qquad B^* \text{ and } P_{component} \text{ are coprime.}$$

$$P_{total} = P = \sum P_{component} = x + 2A - 2 ,$$

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{A} ; \qquad (P_{component} \text{ is odd}) ,$$

$$x = cA + \delta ,$$

$$c = (2m - 3) + \frac{2(l_i + r_i) - \delta}{A} .$$

Note: All components of a given knot do NOT necessarily have the same m-value (see the following examples).

Example:

Given $l_i = 3$; $r_i = 5$; $A = 5$.

Therefore:

$$\delta = |2(l_i + r_i) + \Delta|_A = |2(3 + 5) + 0|_5 = |16|_5 = 1 .$$

$$l_i + r_i = 3 + 5 = 8 = Y + A \qquad \text{thus } Y = 3 \quad \text{and hence also } l_i + r_i = 3 .$$

Therefore:

$$c = (2m^* - 3) + \frac{2(l_i + r_i) - \delta}{A}$$

$$= (2m^* - 3) + \frac{(2 \times 8) - 1}{5} = 2m^* \qquad \text{(hence } c = \text{even)} ,$$

and

$$c = (2m - 3) + \frac{(2 \times 3) - 1}{5} = 2m - 2 \qquad \text{(hence } c = \text{even)} .$$

Therefore:

$$2m^* = 2m - 2 , \qquad \text{hence } m^* = m - 1 .$$

$$x = cA + \delta = 10m - 9 ,$$

$$P_{component} = \frac{10m - 9 + (4 \times 5) - (2 \times 8)}{5} = 2m - 1 \qquad \text{for } l_i + r_i = 8 ,$$

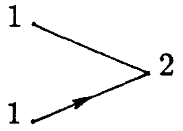
and

$$P_{component} = \frac{10m - 9 + (4 \times 5) - (2 \times 3)}{5} = 2m + 1 \qquad \text{for } l_i + r_i = 3 .$$

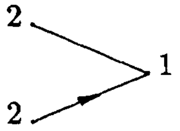
$$P = P_{total} = 10m - 9 + (2 \times 5) - 2 = 10m - 1 .$$

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

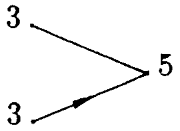
The first-return string-runs of the components are therefore as follows:



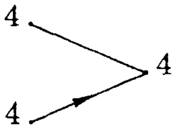
$P_{comp} = 2m + 1\,, \qquad (x = 10m - 9)\,.$



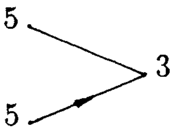
$P_{comp} = 2m + 1\,, \qquad (x = 10m - 9)\,.$



$P_{comp} = 2m - 1\,, \qquad (x = 10m - 9)\,.$



$P_{comp} = 2m - 1\,, \qquad (x = 10m - 9)\,.$



$P_{comp} = 2m - 1\,, \qquad (x = 10m - 9)\,.$

For $m = 2$ we obtain:

$$2m + 1 = 5\,, \quad \text{and}$$
$$2m - 1 = 3\,.$$
$$c = 2m - 2 = 2\,,$$
$$x = 10m - 9 = 11\,.$$
$$P_{component} = 5 \quad \text{and} \quad P_{component} = 3\,,$$
$$P_{total} = 19\,.$$

B^* is coprime to 3 and 5.

(2) — SEMI-STANDARD HERRINGBONE PINEAPPLE KNOTS

$\Delta = 0$; B^* and $P^*_{component}$ have a common divisor greater than 1.

$P^*_{component} = P_{component}$ as calculated in (1).

All derivations are as in (1).

See the Example in (1), where we obtained for $m = 2$:

$$2m + 1 = 5\,, \qquad \text{and}$$
$$2m - 1 = 3\,.$$
$$c = 2m - 2 = 2\,,$$
$$x = 10m - 9 = 11\,.$$
$$P_{component} = 5 \quad \text{and} \quad P_{component} = 3\,.$$

Hence
$$P^*_{component} = 5 \quad \text{and} \quad P^*_{component} = 3\,,$$
$$P_{total} = 19\,.$$

B^* has a divisor of 3 and/or 5 .

(3) — PERFECT HERRINGBONE PINEAPPLE KNOTS

$$\Delta_1 = 1 \qquad \text{(Type 1)}\,.$$
$$\Delta_2 = A - 1 \qquad \text{(Type 2)}\,.$$

B^* and P_{total} are coprime.

The Perfect Herringbone Pineapple Knot has only one component and thus:

$$P_{component} = P_{total} = x + 2A - 2\,.$$

Type 1: $$c = (2m - 3) + \frac{2(l_i + r_i) - \delta + 1}{A}\,,$$

Type 2: $$c = (2m - 3) + \frac{2(l_i + r_i) - \delta - 1}{A}\,.$$

Example 1:

Example 1(a) Type 1 Given $l_i = 3$; $r_i = 4$; $A = 5$.

$$\Delta_1 = 1 \;:\qquad \delta = |2(3+4)+1|_5 = 5\,.$$

$l_i + r_i = 3 + 4 = 7$,

$Y + A = Y + 5 = 7$ therefore $Y = 2$ and hence also $l_i + r_i = 2$.

Therefore:
$$c = (2m - 3) + \frac{(2 \times 7) - 5 + 1}{5} = 2m - 1 \qquad (c = \text{odd})\,,$$

and
$$c = (2m^* - 3) + \frac{(2 \times 2) - 5 + 1}{5} = 2m^* - 3 \qquad (c = \text{odd})\,.$$

Therefore:

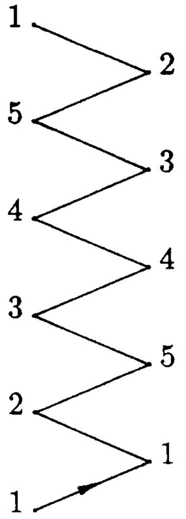
$$2m-1=2m^*-3\,,\qquad \text{hence } m=m^*-1\,.$$

$$x=cA+\delta=10m\,,$$

$$P_{total}=P=10m+8\,.$$

	l_i	r_i		
	5	2	for	$l_i+r_i=7$
	4	3	for	$l_i+r_i=7$
given	**3**	**4**	**for**	$l_i+r_i=7$
	2	5	for	$l_i+r_i=7$
	1	1	for	$l_i+r_i=2$

The first-return string-run is therefore as follows (with $\Delta=1$):



$$x=10m\,,$$
$$P=10m+8\,.$$

Example 1(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (1(a)).

For $l_i=3$, we obtain:

$$r_i=r_{i_2}=|r_{i_1}+\Delta_1|_A=r_i=4+1=5\,.$$

$$l_i+r_i=3+5=8\,,$$

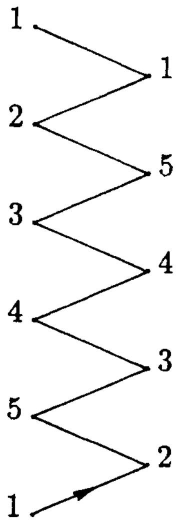
$$Y+A=Y+5=8\qquad \text{therefore } Y=3 \text{ and hence also } l_i+r_i=3\,.$$

$$\Delta_2=5-1=4\,.$$

It is evident that $\delta_{type2}=\delta_{type1}$, and also that $c_{type2}=c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i+r_i=8$
	4	4	for	$l_i+r_i=8$
given	**3**	**5**	**for**	$l_i+r_i=8$
	2	1	for	$l_i+r_i=3$
	1	2	for	$l_i+r_i=3$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 10m\,,$$
$$P = 10m + 8\,.$$

In this example for both types, we obtain when $m = 1$ the following:

$$c = 2m - 1 = 1\,,$$
$$x = 10\,,$$
$$P = 18\,.$$

B^* should not have a divisor greater than 1 in common with 18 .

Therefore: **B^* is coprime with both 2 and 3 .**

Example 2:

Example 2(a) Type 2 Given $l_i = 3$; $r_i = 1$; $A = 5$.

$$\Delta_1 = 1\,, \text{ and so } \Delta_2 = A - 1 = 4\,.$$
$$\delta = |2(3+1)+4|_5 = 2\,.$$

$$l_i + r_i = 3 + 1 = 4 = Y\,,$$

hence also $l_i + r_i = Y + A = 4 + 5 = 9$.

Therefore:

$$c = (2m-3) + \frac{(2\times 9) - 2 - 1}{5} = 2m \qquad (c = \text{even})\,,$$

and

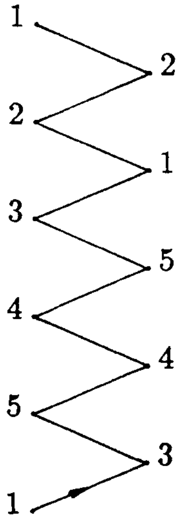
$$c = (2m^*-3) + \frac{(2\times 4) - 2 - 1}{5} = 2m^* - 2 \qquad (c = \text{even})\,.$$

Therefore:

$$2m = 2m^* - 2\,, \qquad \text{hence } m = m^* - 1\,.$$
$$x = cA + \delta = 10m + 2\,,$$
$$P_{total} = P = 10m + 10\,.$$

	l_i	r_i		
	5	4	for	$l_i + r_i = 9$
	4	5	for	$l_i + r_i = 9$
given	**3**	**1**	**for**	$l_i + r_i = 4$
	2	2	for	$l_i + r_i = 4$
	1	3	for	$l_i + r_i = 4$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 10m + 2 \,,$$
$$P = 10m + 10 \,.$$

Example 2(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (2(a)).

For $l_i = 3$, we obtain:

$r_{i_2} = |r_{i_1} - \Delta_2|_A = |r_{i_1} - 4|_5 = 1$, hence $\quad r_{i_1} = r_i = 5$.

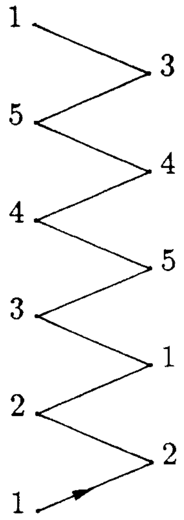
$l_i + r_i = 3 + 5 = 8$,

$Y + A = Y + 5 = 8$ therefore $Y = 3$ and hence also $l_i + r_i = 3$.

As before, it is evident that $\delta_{type1} = \delta_{type2}$, and also that $c_{type1} = c_{type2}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 10m + 2 \,,$$
$$P = 10m + 10 \,.$$

In this example for both types, we obtain when $m = 1$ the following:

$$c = 2m = 2\,,$$
$$x = 12\,,$$
$$P = 20\,.$$

B^* should not have a divisor greater than 1 in common with 20 .

Therefore: **B^* is coprime with both 2 and 5 .**

(4) — SEMI-PERFECT HERRINGBONE PINEAPPLE KNOTS

$$\Delta_1 = 1 \qquad \text{(Type 1)}\,,$$
$$\Delta_2 = A - 1 \qquad \text{(Type 2)}\,.$$

B^* and P_{total} have a common divisor greater than 1.

The Semi-Perfect Herringbone Pineapple Knots are like the Perfect Herringbone Pineapple Knots, except that they require more than one string in their construction.

All formulae derivations are as in **(3)** above.

Example 1 :

Example 1(a) Type 1, and Example 1(b) Type 2

Using the same numerical values for the various parameters as in examples 1(a) and 1(b) above, under **(3)**, we find that the only difference for the Semi-Perfect Herringbone Pineapple Knot is that now B^* has a divisor of 2 and/or 3.

Example 2 :

Example 2(a) Type 2, and Example 2(b) Type 1

Using the same numerical values for the various parameters as in examples 2(a) and 2(b) above, under **(3)**, we find that the only difference for the Semi-Perfect Herringbone Pineapple Knot is that now B^* has a divisor of 2 and/or 5.

(5) — STANDARD BROKEN-HERRINGBONE PINEAPPLE KNOTS

$\Delta = 0\,;$ B^* and $P_{component}$ are coprime.

$$P_{total} = P = \sum P_{component} = x + 2A - 2\,,$$

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{A}\,; \qquad (P_{component} \text{ is even})\,,$$

$$x = cA + \delta\,,$$

$$c \neq (2m - 3) + \frac{2(l_i + r_i) - \delta}{A}\,.$$

Note: All components of a given knot do NOT necessarily have the same m-value (see the following examples).

Example:

Given $l_i = 3$; $r_i = 5$; $A = 5$.

Therefore:

$$\delta = |2(l_i + r_i) + \Delta|_A = |2(3+5)+0|_5 = |16|_5 = 1 .$$

$$l_i + r_i = 3 + 5 = 8 \qquad \text{hence } Y = 3 \quad \text{and hence also } l_i + r_i = 3 .$$

Therefore:

$$c \neq (2m - 3) + \frac{2(l_i + r_i) - \delta}{A} ,$$

$$c \neq (2m - 3) + \frac{(2 \times 8) - 1}{5} ,$$

$$c \neq 2m \qquad \text{hence} \quad c = \text{odd} ,$$

and

$$c \neq (2m^* - 3) + \frac{(2 \times 3) - 1}{5} ,$$

$$c \neq 2m^* - 2 \qquad \text{hence} \quad c = \text{odd} .$$

Therefore:

$$c = 2m - 1 ,$$

$$x = cA + \delta = 10m - 4 ,$$

$$P_{component} = \frac{10m - 4 + (4 \times 5) - (2 \times 8)}{5} = 2m \qquad \text{for} \quad l_i + r_i = 8 ,$$

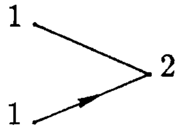
and

$$P_{component} = \frac{10m - 4 + (4 \times 5) - (2 \times 3)}{5} = 2m + 2 \qquad \text{for} \quad l_i + r_i = 3 ,$$

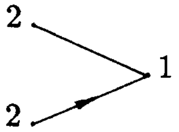
$$P = P_{total} = 10m - 4 + (2 \times 5) - 2 = 10m + 4 .$$

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

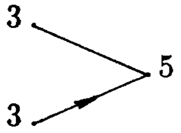
The first-return string-runs of the components are therefore as follows:



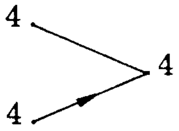
$$P_{comp} = 2m + 2\,, \qquad (x = 10m - 4)\,.$$



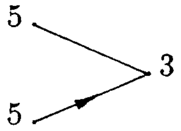
$$P_{comp} = 2m + 2\,, \qquad (x = 10m - 4)\,.$$



$$P_{comp} = 2m\,, \qquad (x = 10m - 4)\,.$$



$$P_{comp} = 2m\,, \qquad (x = 10m - 4)\,.$$



$$P_{comp} = 2m\,, \qquad (x = 10m - 4)\,.$$

For $m = 2$ we obtain:

$$2m + 2 = 6\,, \qquad \text{and}$$
$$2m = 4\,.$$
$$c = 2m - 1 = 3\,,$$
$$x = 10m - 4 = 16\,.$$
$$P_{component} = 6 \quad \text{and} \quad P_{component} = 4\,,$$
$$P_{total} = 24\,.$$

B^* is coprime to 2 and 3.

(6) — SEMI-STANDARD BROKEN-HERRINGBONE PINEAPPLE KNOTS

$\Delta = 0$; B^* and $P^*_{component}$ have a common divisor greater than 1.

$P^*_{component} = P_{component}$ as calculated in (**5**).

All derivations are as in (**5**).

See the Example in (**5**), where we obtained for $m = 2$:

$$2m = 4\,, \qquad \text{and}$$
$$2m + 2 = 6\,.$$
$$c = 2m - 1 = 3\,,$$
$$x = 10m - 4 = 16\,.$$
$$P_{component} = 6 \quad \text{and} \quad P_{component} = 4\,.$$

Hence $$P^*_{component} = 6 \quad \text{and} \quad P^*_{component} = 4\,,$$
$$P_{total} = 24\,.$$

B^* has a divisor of 2 and/or 3 .

(7) — PERFECT BROKEN-HERRINGBONE PINEAPPLE KNOTS

Δ can take any value in the range $1, \ldots, (A-1)$ which is coprime with A .

B^* and $P = P_{total}$ are coprime.

Type 1 : $\Delta_1 = \epsilon \qquad 1 \le \epsilon < A/2\,,$

Type 2 : $\Delta_2 = A - \epsilon \qquad 1 \le \epsilon < A/2\,.$

Thus: ϵ and A are coprime.

The Perfect Broken-Herringbone Pineapple Knot has one component only, and hence

$$P_{component} = P_{total} = P = x + 2A - 2\,.$$

For $\epsilon = 1$:
$$c \neq (2m-3) + \frac{2(l_i + r_i) - \delta + 1}{A} \qquad \text{(Type 1)}\,,$$
$$c \neq (2m-3) + \frac{2(l_i + r_i) - \delta - 1}{A} \qquad \text{(Type 2)}\,.$$

Example 1 :

Example 1(a) Type 1 Given $\epsilon = 1$; $l_i = 3$; $r_i = 4$; $A = 5$.

$\epsilon = 1$ implies $\Delta_1 = 1$: $\delta = |2(3+4)+1|_5 = 5$.

$l_i + r_i = 3 + 4 = 7$,

$Y + A = Y + 5 = 7$ therefore $Y = 2$ and hence also $l_i + r_i = 2$.

Therefore:

$$\begin{aligned} c &\neq (2m-3) + \frac{(2\times 7)-5+1}{5} \\ &\neq 2m-1 \qquad\qquad (c = \text{even})\,, \end{aligned}$$

and

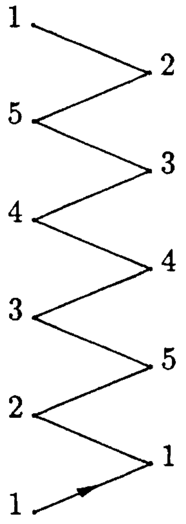
$$\begin{aligned} c &\neq (2m^*-3) + \frac{(2\times 2)-5+1}{5}\,, \\ &\neq 2m^*-3 \qquad\qquad (c = \text{even})\,. \end{aligned}$$

Therefore:

$$\begin{aligned} c &= 2m\,, \\ x &= cA + \delta = 10m+5\,, \\ P_{total} &= P = 10m+13\,. \end{aligned}$$

	l_i	r_i		
	5	2	for	$l_i + r_i = 7$
	4	3	for	$l_i + r_i = 7$
given	**3**	**4**	**for**	$l_i + r_i = 7$
	2	5	for	$l_i + r_i = 7$
	1	1	for	$l_i + r_i = 2$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 10m+5\,,$$
$$P = 10m+13\,.$$

Example 1(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (1(a)).

For $l_i = 3$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} + \Delta_1|_A = |4+1|_5 = 5\,.$$

$$l_i + r_i = 3 + 5 = 8\,,$$

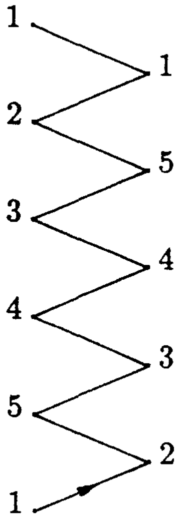
$Y + A = Y + 5 = 8$ therefore $Y = 3$ and hence also $l_i + r_i = 3$.

$$\Delta_2 = 5 - 1 = 4\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 10m + 5\,,$$
$$P = 10m + 13\,.$$

In this example for both Types, we obtain when $m = 1$ the following:

$$c = 2m = 2\,,$$
$$x = 15\,,$$
$$P = 23\,.$$

Since B^* and P are coprime, B^* must be coprime with 23.

Example 2:

Example 2(a) Type 1 Given $\epsilon = 2$; $l_i = 2$; $r_i = 4$; $A = 5$.

$$\epsilon = 2 \text{ implies } \Delta_1 = 2 : \qquad \delta = |2(2+4)+2|_5 = 4\,.$$

$$l_i + r_i = 2 + 4 = 6 = A + 1\,,$$

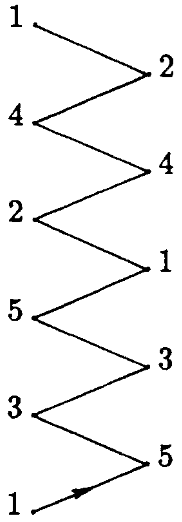
Hence for all $l_i + r_i$ we obtain the value 6.

$$x = cA + \delta = 5c + 4\,,$$

$$P_{total} = P = x + 2A - 2 = 5c + 12\,.$$

	l_i	r_i		
	5	1	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	1	5	for	$l_i + r_i = 6$

The first-return string-run is therefore as follows (with $\Delta = 2$):



$$x = 5c + 4\,,$$
$$P = 5c + 12\,.$$

Example 2(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (2(a)).

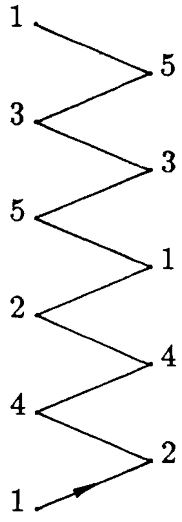
For $l_i = 2$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} + \Delta_1|_A = |4 + 2|_5 = 1\,.$$
$$l_i + r_i = 2 + 1 = 3\,,$$
$$Y = 3 \qquad \text{therefore } Y + A = 8 \text{ and hence also } l_i + r_i = 8\,.$$
$$\Delta_2 = 5 - 2 = 3\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
given	**2**	**1**	**for**	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

The first-return string-run is therefore as follows (with $\Delta = 3$):



$$x = 5c + 4\,,$$
$$P = 5c + 12\,.$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 14\,,$$
$$P = 22\,.$$

Since B^* and P are coprime, B^* must be coprime with 2 and 11 .

Example 3 :

Example 3(a) Type 2 Given $\epsilon = 2$; $l_i = 4$; $r_i = 1$; $A = 5$.

$\epsilon = 2$ implies $\Delta_2 = A - \epsilon = 3$: $\delta = |2(4+1)+3|_5 = 3$.

$l_i + r_i = 4 + 1 = 5 = Y$,

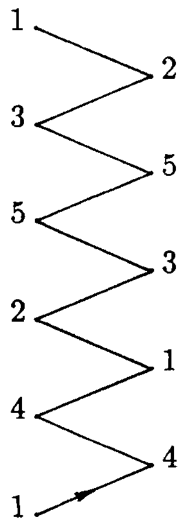
hence also $l_i + r_i = Y + A = 5 + 5 = 10$.

$x = cA + \delta = 5c + 3$,

$P_{total} = P = 5c + 11$.

	l_i	r_i		
	5	5	for	$l_i + r_i = 10$
given	**4**	**1**	**for**	$l_i + r_i = 5$
	3	2	for	$l_i + r_i = 5$
	2	3	for	$l_i + r_i = 5$
	1	4	for	$l_i + r_i = 5$

The first-return string-run is therefore as follows (with $\Delta = 3$):



$x = 5c + 3$,
$P = 5c + 11$.

Example 3(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (3(a)).

For $l_i = 4$, we obtain:

$r_{i_2} = |r_{i_1} - \Delta_2|_A = |r_{i_1} - 3|_5 = 1$, hence $r_{i_1} = r_i = 4$.

$\Delta_1 = \epsilon = A - \Delta_2 = 5 - 3 = 2$.

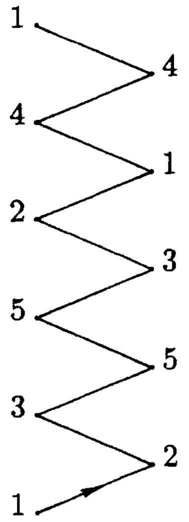
$l_i + r_i = 4 + 4 = 8$,

$Y + A = Y + 5 = 8$ therefore $Y = 3$ and hence also $l_i + r_i = 3$.

As before, it is evident that $\delta_{type1} = \delta_{type2}$, and also that $c_{type1} = c_{type2}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
given	**4**	**4**	**for**	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

The first-return string-run is therefore as follows (with $\Delta = 2$):



$$x = 5c + 3\ ,$$
$$P = 5c + 11\ .$$

In this example for both Types, we obtain when $c = 1$ the following:

$$x = 8\ ,$$
$$P = 16\ .$$

B^* should not have a divisor greater than 1 in common with 16 ,

Therefore: **B^* is coprime with 2 ; that is, B^* is odd .**

Example 4 :

Example 4(a) Type 2 Given $\epsilon = 1$; $l_i = 4$; $r_i = 5$; $A = 5$.

$\epsilon = 1$ implies $\Delta_2 = A - \epsilon = 4$: $\delta = |2(4+5)+4|_5 = 2$.

$l_i + r_i = 4 + 5 = 9$,

$Y + A = Y + 5 = 9$ therefore $Y = 4$ and hence also $l_i + r_i = 4$.

Therefore:

$$\begin{aligned} c &\neq (2m-3) + \frac{(2\times 9)-2-1}{5} \\ &\neq 2m \qquad\qquad (c = \text{odd})\ , \end{aligned}$$

and

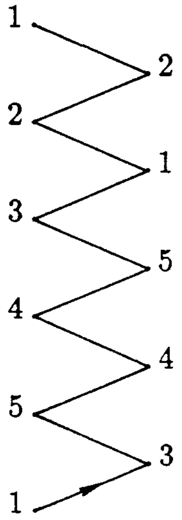
$$\begin{aligned} c &\neq (2m^*-3) + \frac{(2\times 4)-2-1}{5} \\ &\neq 2m^* - 2 \qquad\qquad (c = \text{odd})\ . \end{aligned}$$

Therefore:

$$\begin{aligned} c &= 2m - 1\,, \\ x &= cA + \delta = 10m - 3\,, \\ P_{total} &= P = 10m + 5\,. \end{aligned}$$

	l_i	r_i		
	5	4	for	$l_i + r_i = 9$
given	**4**	**5**	**for**	$l_i + r_i = 9$
	3	1	for	$l_i + r_i = 4$
	2	2	for	$l_i + r_i = 4$
	1	3	for	$l_i + r_i = 4$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$\begin{aligned} x &= 10m - 3\,, \\ P &= 10m + 5\,. \end{aligned}$$

Example 4(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (4(a)).

For $l_i = 4$, we obtain:

$$r_i = r_{i_1} = |r_{i_2} + \Delta_2|_A = |5 + 4|_5 = 4\,.$$

$$l_i + r_i = 4 + 4 = 8\,,$$

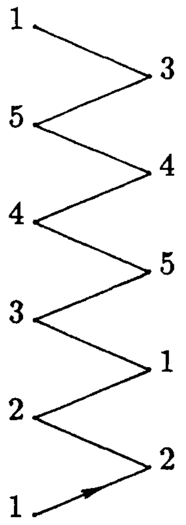
$$Y + A = Y + 5 = 8 \qquad \text{therefore } Y = 3 \text{ and hence also } l_i + r_i = 3\,,$$

$$\Delta_1 = \epsilon = A - \Delta_2 = 1\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	5	3	for	$l_i + r_i = 8$
given	**4**	**4**	**for**	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
	2	1	for	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 10m - 3\,,$$
$$P = 10m + 5\,.$$

In this example for both types, we obtain when $m = 2$ the following:

$$c = 2m - 1 = 3\,,$$
$$x = 17\,,$$
$$P = 25\,.$$

Since B^* and P are coprime, B^* must be coprime with 5 .

(8) — SEMI-PERFECT BROKEN-HERRINGBONE PINEAPPLE KNOTS

Δ can take any value in the range $1, \ldots, (A-1)$ which is coprime with A .

B^* and $P = P_{total}$ have a common divisor greater than 1 .

The Semi-Perfect Broken-Herringbone Pineapple Knots are like the Perfect Broken-Herringbone Pineapple Knots, except that they require more than one string in their construction.

All formulae derivations are as in (7) above.

Example 1 :

Example 1(a) Type 1, and Example 1(b) Type 2

Using the same numerical values for the various parameters as in examples 1(a) and 1(b) under (7), we find that the only difference for the Semi-Perfect Broken-Herringbone Pineapple Knot is that now B^* and P must have a common divisor of 23.

Example 2 :

Example 2(a) Type 1, and Example 2(b) Type 2

Using the same numerical values for the various parameters as in examples 2(a) and 2(b) above, under (7), we find that the only difference for the Semi-Perfect Broken-Herringbone Pineapple Knot is that now B^* and P must have a common divisor of 2 and/or 11.

Example 3:

Example 3(a) Type 2, and Example 3(b) Type 1

Using the same numerical values for the various parameters as in examples 3(a) and 3(b) under (7), we find that the only difference for the Semi-Perfect Broken-Herringbone Pineapple Knot is that now B^* and P must have a common divisor of 2, hence B^* is even.

Example 4:

Example 4(a) Type 2, and Example 4(b) Type 1

Using the same numerical values for the various parameters as in examples 4(a) and 4(b) under (7), we find that the only difference for the Semi-Perfect Broken-Herringbone Pineapple Knot is that now B^* and P must have a common divisor of 5.

(9) — COMPOUND BROKEN-HERRINGBONE PINEAPPLE KNOTS

For knots in this sub-class, Δ can take any value in the range $2, \ldots, (A-2)$ which has with A a common divisor.

B^* and $P_{component}$ are coprime.

The number of passes is the same for each component of a given knot.

We define the number of passes per component by α. Hence the number of components of a given knot is A/α .

When a component is completed, α bight-boundaries on each edge of the knot have been visited, and then the equation $|1 + \alpha \cdot \Delta|_A = 1$ is satisfied. This implies that $\alpha \cdot \Delta$ is an integral multiple (say σ) of A ; that is $\alpha \cdot \Delta = \sigma \cdot A$; and so, finally, we can write:

$$\Delta = \frac{\sigma \cdot A}{\alpha} .$$

We may note that σ and α are coprime.

Type 1 : $\Delta_1 = \epsilon$ $\quad 2 \leq \epsilon < A/2$,

Type 2 : $\Delta_2 = A - \epsilon$ $\quad 2 \leq \epsilon < A/2$.

Note that Type 1 and Type 2 have the same Δ-value when $\Delta_1 = \Delta_2 = A/2$.

Furthermore, since Δ and A have a common divisor greater than 1 , ϵ and A are not coprime.

$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \ldots ,$$

$$P_{component} = P_\alpha = \alpha(c+4) + \left\{ \frac{\alpha \cdot \delta - 2(\Sigma l_i + \Sigma r_i)}{A} \right\} ,$$

$$P_{total} = \Sigma P_{component} = x + 2A - 2 .$$

Example 1:

Given $l_i = 2$; $r_i = 1$; $A = 6$.

$A = 6$ has the divisors 2 and 3; thus $\alpha = 2$ or 3.

Let α be equal to 2; then $\Delta = \sigma A/2 = 3\sigma$.

But since $2 \leq \Delta \leq A - 2$, we obtain $\sigma = 1$ and hence $\Delta = 3$.

Example 1(a)

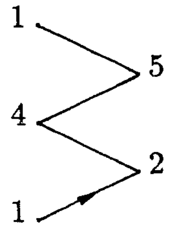
$$\begin{aligned} \delta &= |2(2+1)+3|_6 = 3\,, \\ x &= cA + \delta = 6c+3\,, \\ P &= P_{total} = x + 2A - 2 = 6c + 13\,. \end{aligned}$$

$$l_i + r_i = 2 + 1 = 3\,,$$

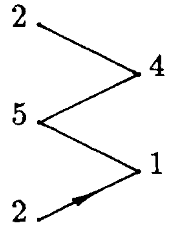
$$Y = 3 \qquad \text{hence also } l_i + r_i = 3 + A = 3 + 6 = 9\,.$$

	l_i	r_i		
	6	3	for	$l_i + r_i = 9$
	5	4	for	$l_i + r_i = 9$
	4	5	for	$l_i + r_i = 9$
	3	6	for	$l_i + r_i = 9$
given	**2**	**1**	**for**	$l_i + r_i = 3$
	1	2	for	$l_i + r_i = 3$

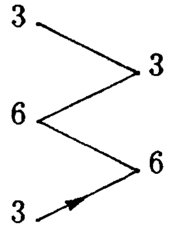
The first-return string-runs are therefore as follows (with $\Delta = 3$):



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(1+4+2+5)}{6} = 2c + 5\,\cdot$$



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(2+5+1+4)}{6} = 2c + 5\,\cdot$$



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(3+6+6+3)}{6} = 2c + 3\,\cdot$$

Example 1(b)

We now give details of the mirror-image of the foregoing example (1(a)).

For $l_i = 2$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} + \Delta_1|_A = |1+3|_6 = 4\,.$$

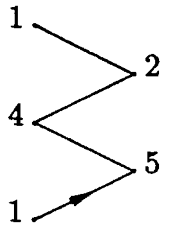
$$l_i + r_i = 2 + 4 = 6\,,$$

$$Y = 6 \qquad \text{hence also } l_i + r_i = 6 + A = 6 + 6 = 12\,.$$

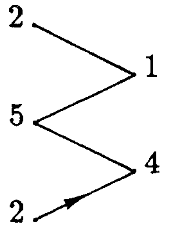
$$\Delta_2 = A - \Delta_1 = 6 - 3 = 3 = \Delta_1\,.$$

	l_i	r_i		
	6	6	for	$l_i + r_i = 12$
	5	1	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	1	5	for	$l_i + r_i = 6$

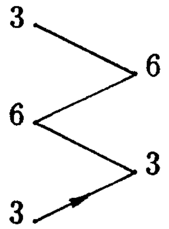
The first-return string-runs are therefore as follows (with $\Delta = 3$):



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(1+4+5+2)}{6} = 2c + 5 \ .$$



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(2+5+4+1)}{6} = 2c + 5 \ .$$



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(3+6+3+6)}{6} = 2c + 3 \ .$$

It is of course inconsequential which of the above two knots 1(a) or 1(b) is called Type 1 or Type 2. We shall however indicate the knots which have a half-cycle ${}_1\nearrow^1$; ${}_1\nearrow^2$ and ${}_1\nearrow^3$ as Type 1 knots and hence the knots which have a half-cycle ${}_1\nearrow^4$; ${}_1\nearrow^5$ and ${}_1\nearrow^6$ as Type 2 knots.

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 3 = 15 \ ,$$
$$P_{total} = P = 6c + 13 = 25 \ ,$$
$$P_{component} = 9 \text{ and } P_{component} = 7 \ .$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 3 and 7 .

Example 2:

Given $l_i = 2$; $r_i = 2$; $A = 6$.

Let α be equal to 3 ; then $\Delta = \frac{\sigma A}{3} = 2\sigma$.
Since $2 \leq \Delta \leq A - 2$, hence $2 \leq \Delta \leq 4$, the value of σ can be 1 or 2.
For $\sigma = 1$ we obtain $\Delta = 2$, and for $\sigma = 2$ we obtain $\Delta = 4$.

Example 2(a) Type 1

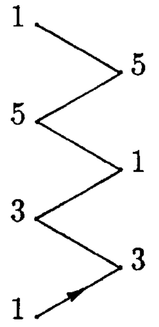
$$\Delta = 2\,.$$

$$\begin{aligned} \delta &= |2(2+2)+2|_6 = 4\,, \\ x &= cA + \delta = 6c + 4\,, \\ P &= P_{total} = x + 2A - 2 = 6c + 14\,. \end{aligned}$$

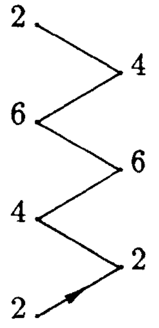
$l_i + r_i = 2 + 2 = 4$,
$Y = 4$ hence also $l_i + r_i = 4 + A = 4 + 6 = 10$.

	l_i	r_i		
	6	4	for	$l_i + r_i = 10$
	5	5	for	$l_i + r_i = 10$
	4	6	for	$l_i + r_i = 10$
	3	1	for	$l_i + r_i = 4$
given	**2**	**2**	**for**	$l_i + r_i = 4$
	1	3	for	$l_i + r_i = 4$

The first-return string-runs are therefore as follows (with $\Delta = 2$):



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(1+3+5+3+1+5)}{6} = 3c + 8\,.$$



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(2+4+6+2+6+4)}{6} = 3c + 6\,.$$

Example 2(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (2(a)).

For $l_i = 2$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} + \Delta_1|_A = |2+2|_6 = 4 .$$

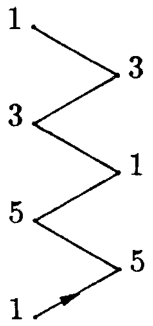
$$l_i + r_i = 2 + 4 = 6 ,$$

$$Y = 6 \qquad \text{hence also } l_i + r_i = 6 + A = 6 + 6 = 12 .$$

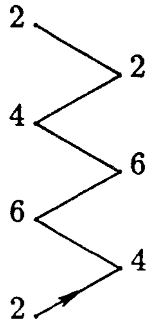
$$\Delta_2 = A - \Delta_1 = 6 - 2 = 4 .$$

	l_i	r_i		
	6	6	for	$l_i + r_i = 12$
	5	1	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	1	5	for	$l_i + r_i = 6$

The first-return string-runs are therefore as follows (with $\Delta = 4$):



$$P_{component} = 3(c+4) + \frac{(3\times 4) - 2(1+5+3+5+1+3)}{6} = 3c + 8 \cdot$$



$$P_{component} = 3(c+4) + \frac{(3\times 4) - 2(2+6+4+4+6+2)}{6} = 3c + 6 \cdot$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 4 = 16 ,$$

$$P_{total} = P = 6c + 14 = 26 ,$$

$$P_{component} = 14 \text{ and } P_{component} = 12 .$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 2 , 3 and 7 .

Example 3 :

Given $l_i = 2\ ;\ r_i = 6\ ;\ A = 6$.

Let α be equal to 3; then $\Delta = \dfrac{\sigma A}{3} = 2\sigma$.

Since $2 \leq \Delta \leq A - 2$, hence $2 \leq \Delta \leq 4$, the value of σ can be 1 or 2.

For $\sigma = 1$ we obtain $\Delta = 2$, and for $\sigma = 2$ we obtain $\Delta = 4$.

Example 3(a) Type 2

$$\Delta = 4\,.$$

Note that we are dealing with a Type 2 since $\Delta = 4$ and hence $\epsilon = 2$ (Type 2: $\Delta_2 = A - \epsilon$ with $2 \leq \epsilon < A/2$).

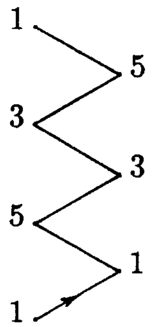
$$\delta = |2(2+6)+4|_6 = 2\,,$$
$$x = cA + \delta = 6c + 2\,,$$
$$P = P_{total} = x + 2A - 2 = 6c + 12\,.$$

$l_i + r_i = 2 + 6 = 8$,

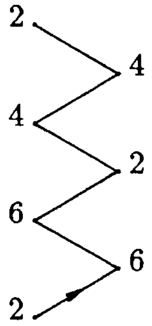
$Y + A = Y + 6 = 8$ therefore $Y = 2$ and hence also $l_i + r_i = 2$.

	l_i	r_i		
	6	2	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
given	**2**	**6**	**for**	$l_i + r_i = 8$
	1	1	for	$l_i + r_i = 2$

The first-return string-runs are therefore as follows (with $\Delta = 4$):



$$P_{component} = 3(c+4) + \frac{(3\times 2) - 2(1+5+3+1+3+5)}{6} = 3c + 7\ \cdot$$



$$P_{component} = 3(c+4) + \frac{(3\times 2) - 2(2+6+4+6+2+4)}{6} = 3c + 5\ \cdot$$

Example 3(b) Type 1

We now give details of the mirror-image of the foregoing example (3(a) Type 2).

For $l_i = 2$, we obtain :

$$r_i = r_{i_1} = |r_{i_2} + \Delta_2|_A = |6 + 4|_6 = 4 .$$

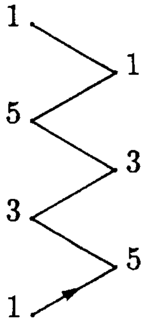
$$l_i + r_i = 2 + 4 = 6 ,$$

$$Y = 6 \qquad \text{hence also } l_i + r_i = 6 + A = 6 + 6 = 12 .$$

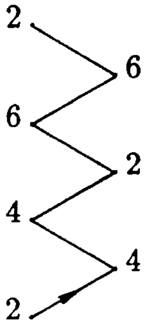
$$\Delta_1 = A - \Delta_2 = 6 - 4 = 2 .$$

	l_i	r_i		
	6	6	for	$l_i + r_i = 12$
	5	1	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	1	5	for	$l_i + r_i = 6$

The first-return string-runs are therefore as follows (with $\Delta = 2$) :



$$P_{component} = 3(c + 4) + \frac{(3 \times 2) - 2(1 + 3 + 5 + 5 + 3 + 1)}{6} = 3c + 7 \cdot$$



$$P_{component} = 3(c + 4) + \frac{(3 \times 2) - 2(2 + 4 + 6 + 4 + 2 + 6)}{6} = 3c + 5 \cdot$$

In this example for both Types, we obtain when $c = 2$ the following :

$$x = 6c + 2 = 14 ,$$

$$P_{total} = P = 6c + 12 = 24 ,$$

$$P_{component} = 13 \text{ and } P_{component} = 11 .$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 13 and 11 .

(10) — SEMI-COMPOUND BROKEN-HERRINGBONE PINEAPPLE KNOTS

For knots in this sub-class, Δ can take any value in the range $2, \ldots, (A-2)$ which has with A a common divisor.

B^* and $P^*_{component}$ have a common divisor greater than 1.

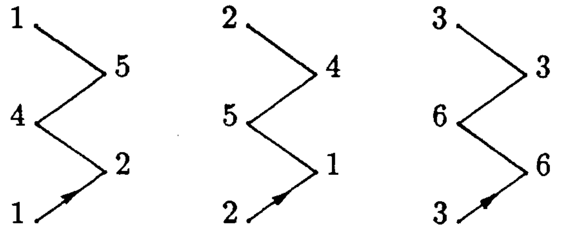
$P^*_{component} = P_{component}$ as calculated in (**9**).

All derivations are as in (**9**).

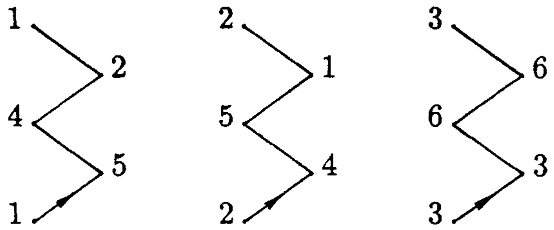
See the Examples in (**9**) :

Example 1 :

Example 1(a) and 1(b)



and



$P^*_{comp} =$ $2c+5$ $\quad$ $P^*_{comp} =$ $2c+5$ $\quad$ $P^*_{comp} =$ $2c+3$ $\quad$ $P^*_{comp} =$ $2c+5$ $\quad$ $P^*_{comp} =$ $2c+5$ $\quad$ $P^*_{comp} =$ $2c+3$

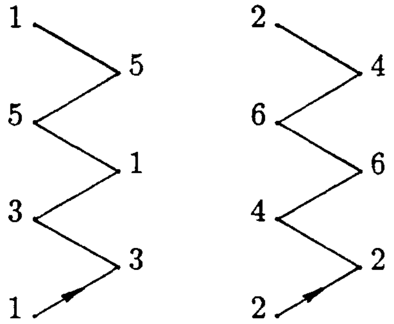
For $c = 2$ we obtain :

$$x = 6c + 3 = 15\,,$$
$$P_{total} = P = 6c + 13 = 25\,,$$
$$P^*_{component} = 9 \text{ and } P^*_{component} = 7\,.$$

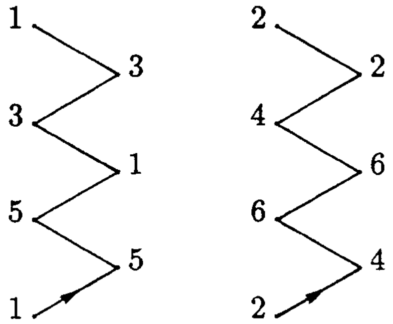
B^* and $P^*_{component}$ have a common divisor of 3 and/or 7 .

Example 2 :

Example 2(a) and 2(b)



and



$P^*_{comp} =$ $3c+8$ $\quad$ $P^*_{comp} =$ $3c+6$ $\quad$ $P^*_{comp} =$ $3c+8$ $\quad$ $P^*_{comp} =$ $3c+6$

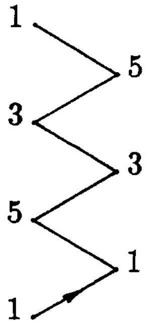
For $c = 2$ we obtain:

$$x = 6c + 4 = 16\,,$$
$$P_{total} = P = 6c + 14 = 26\,,$$
$$P^*_{component} = 14 \text{ and } P^*_{component} = 12\,.$$

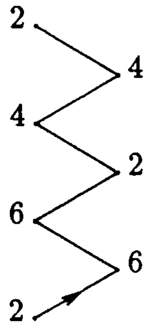
B^* and $P^*_{component}$ have a common divisor of 2 and/or 3 and/or 7.

Example 3:

Example 3(a) and 3(b)

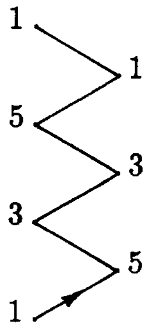


$P^*_{comp} =$ $3c + 7$

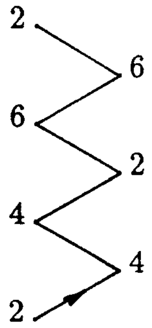


$P^*_{comp} =$ $3c + 5$

and



$P^*_{comp} =$ $3c + 7$



$P^*_{comp} =$ $3c + 5$

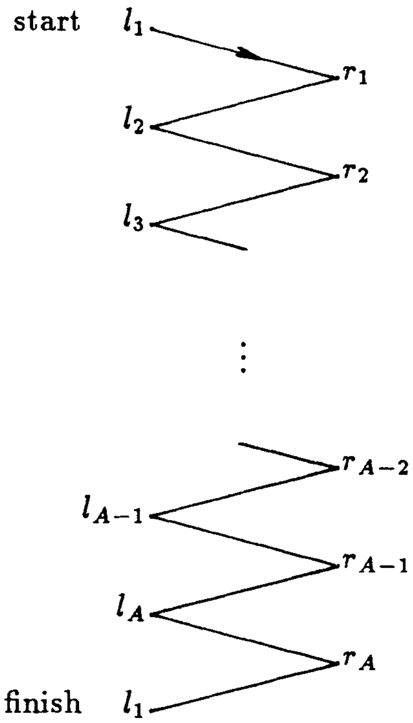
For $c = 2$ we obtain:

$$x = 6c + 2 = 14\,,$$
$$P_{total} = P = 6c + 12 = 24\,,$$
$$P^*_{component} = 13 \text{ and } P^*_{component} = 11\,.$$

B^* and $P^*_{component}$ have a common divisor of 11 and/or 13.

(II)

THE FORMULAE FOR BRAIDING DOWNWARDS



$$\begin{array}{rlrl}
l_2 &= |l_1 - \Delta|_A & r_2 &= |r_1 + \Delta|_A \\
l_3 &= |l_2 - \Delta|_A & r_3 &= |r_2 + \Delta|_A \\
l_4 &= |l_3 - \Delta|_A & r_4 &= |r_3 + \Delta|_A \\
&\vdots & &\vdots \\
l_{i+1} &= |l_i - \Delta|_A & r_{i+1} &= |r_i + \Delta|_A \\
&\vdots & &\vdots
\end{array}$$

In the above, $i = 1, 2, 3, \ldots, (A-1)$ and
Δ is an integer belonging to the set : $\{0, 1, 2, 3, \ldots, (A-2), (A-1)\}$.

All components of a given knot have the same Δ-value.

$\delta = |2(l_i + r_i) - \Delta|_A$ where δ is an integer belonging to the set : $\{1, 2, 3, \ldots, (A-1), A\}$.

All components of a given knot have the same δ-value.

For all half-cycles of a given knot, the value of $(l_i + r_i)$ is either equal to :
(1) $A+1$
or
(2) Y, or $Y + A$, where $2 \leq Y \leq A$.

All components of a given knot have the same x-value, and since $x = cA + \delta$ it follows that all components of a given knot have the same c-value.

In general we have :

Type 1 : $\Delta_1 = \epsilon$ for $1 \leq \epsilon < A/2$,
Type 2 : $\Delta_2 = A - \epsilon$ for $1 \leq \epsilon < A/2$.
Thus $|l_{i_2} + r_{i_2}|_A = |l_{i_1} + r_{i_1} - \Delta_1|_A$.

Note that Type 1 and Type 2 have the same Δ-value when $\Delta_1 = \Delta_2 = A/2$.

For $l_{i_1} = l_{i_2}$ the following relationship holds :

$$r_{i_2} = |r_{i_1} - \Delta_1|_A = |r_{i_1} - A + \Delta_2|_A = |r_{i_1} + \Delta_2|_A .$$

Note the special case when $\Delta = 0$, in which case Type 1 is identical to Type 2; and also the special case when $\Delta_1 = \Delta_2 = A/2$, in which case:

$$r_{i_2} = |r_{i_1} - A/2|_A = |r_{i_1} + A/2|_A .$$

(1) — STANDARD HERRINGBONE PINEAPPLE KNOTS

$$\Delta = 0 ; \qquad B^* \text{ and } P_{component} \text{ are coprime.}$$

$$P_{total} = P = \sum P_{component} = x + 2A - 2 ,$$

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{A} ; \qquad (P_{component} \text{ is odd}) ,$$

$$x = cA + \delta ,$$

$$c = (2m - 3) + \frac{2(l_i + r_i) - \delta}{A} .$$

Note: All components of a given knot do NOT necessarily have the same m-value (see the following examples).

Example:

Given $l_i = 3$; $r_i = 5$; $A = 5$.

Therefore:

$$\delta = |2(l_i + r_i) - \Delta|_A = |2(3 + 5) - 0|_5 = |16|_5 = 1 .$$

$$l_i + r_i = 3 + 5 = 8 = Y + A \qquad \text{thus } Y = 3 \quad \text{and hence also } l_i + r_i = 3 .$$

Therefore:

$$c = (2m^* - 3) + \frac{2(l_i + r_i) - \delta}{A}$$

$$= (2m^* - 3) + \frac{(2 \times 8) - 1}{5} = 2m^* \qquad \text{(hence } c = \text{even)} ,$$

and

$$c = (2m - 3) + \frac{(2 \times 3) - 1}{5} = 2m - 2 \qquad \text{(hence } c = \text{even)} .$$

Therefore:

$$2m^* = 2m - 2 , \qquad \text{hence } m^* = m - 1 .$$

$$x = cA + \delta = 10m - 9 ,$$

$$P_{component} = \frac{10m - 9 + (4 \times 5) - (2 \times 8)}{5} = 2m - 1 \qquad \text{for } l_i + r_i = 8 ,$$

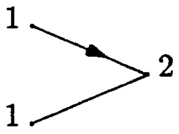
and

$$P_{component} = \frac{10m - 9 + (4 \times 5) - (2 \times 3)}{5} = 2m + 1 \qquad \text{for } l_i + r_i = 3 .$$

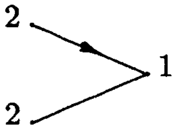
$$P = P_{total} = 10m - 9 + (2 \times 5) - 2 = 10m - 1 .$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

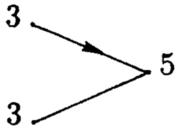
The first-return string-runs of the components are therefore as follows:



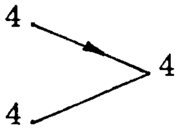
$$P_{comp} = 2m + 1\,, \qquad (x = 10m - 9)\,.$$



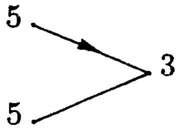
$$P_{comp} = 2m + 1\,, \qquad (x = 10m - 9)\,.$$



$$P_{comp} = 2m - 1\,, \qquad (x = 10m - 9)\,.$$



$$P_{comp} = 2m - 1\,, \qquad (x = 10m - 9)\,.$$



$$P_{comp} = 2m - 1\,, \qquad (x = 10m - 9)\,.$$

For $m = 2$ we obtain:

$$2m + 1 = 5\,, \quad \text{and}$$
$$2m - 1 = 3\,.$$
$$c = 2m - 2 = 2\,,$$
$$x = 10m - 9 = 11\,.$$
$$P_{component} = 5 \quad \text{and} \quad P_{component} = 3\,,$$
$$P_{total} = 19\,.$$

B^* is coprime to 3 and 5.

(2) — SEMI-STANDARD HERRINGBONE PINEAPPLE KNOTS

$\Delta = 0$; B^* and $P^*_{component}$ have a common divisor greater than 1.

$P^*_{component} = P_{component}$ as calculated in **(1)**.

All derivations are as in **(1)**.

See the Example in **(1)**, where we obtained for $m = 2$:

$$2m + 1 = 5 , \quad \text{and}$$
$$2m - 1 = 3 .$$
$$c = 2m - 2 = 2 ,$$
$$x = 10m - 9 = 11 .$$
$$P_{component} = 5 \quad \text{and} \quad P_{component} = 3 .$$

Hence

$$P^*_{component} = 5 \quad \text{and} \quad P^*_{component} = 3 ,$$
$$P_{total} = 19 .$$

B^* has a divisor of 3 and/or 5 .

(3) — PERFECT HERRINGBONE PINEAPPLE KNOTS

$$\Delta_1 = 1 \qquad \text{(Type 1)} .$$
$$\Delta_2 = A - 1 \qquad \text{(Type 2)} .$$

B^* and P_{total} are coprime.

The Perfect Herringbone Pineapple Knot has only one component and thus:

$$P_{component} = P_{total} = x + 2A - 2 .$$

Type 1: $$c = (2m - 3) + \frac{2(l_i + r_i) - \delta - 1}{A} ,$$

Type 2: $$c = (2m - 3) + \frac{2(l_i + r_i) - \delta + 1}{A} .$$

Example 1:

Example 1(a) Type 1 Given $l_i = 3$; $r_i = 5$; $A = 5$.

$$\Delta_1 = 1 \ : \qquad \delta = |2(3 + 5) - 1|_5 = 5 .$$
$$l_i + r_i = 3 + 5 = 8 ,$$

$Y + A = Y + 5 = 8$ therefore $Y = 3$ and hence also $l_i + r_i = 3$.

Therefore:

$$c = (2m - 3) + \frac{(2 \times 8) - 5 - 1}{5} = 2m - 1 \qquad (c = \text{odd}) ,$$

and

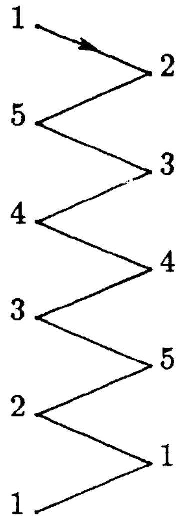
$$c = (2m^* - 3) + \frac{(2 \times 3) - 5 - 1}{5} = 2m^* - 3 \qquad (c = \text{odd}) .$$

Therefore:

$$2m - 1 = 2m^* - 3\,, \qquad \text{hence } m = m^* - 1\,.$$
$$x = cA + \delta = 10m\,,$$
$$P_{total} = P = 10m + 8\,.$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 10m\,,$$
$$P = 10m + 8\,.$$

Example 1(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (1(a)).

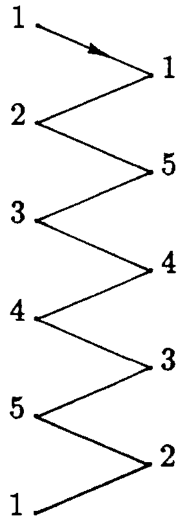
For $l_i = 3$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |5 - 1|_5 = 4\,.$$
$$l_i + r_i = 3 + 4 = 7\,,$$
$$Y + A = Y + 5 = 7 \qquad \text{therefore } Y = 2 \text{ and hence also } l_i + r_i = 2\,.$$
$$\Delta_2 = 5 - 1 = 4\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	1	1	for	$l_i + r_i = 2$
	2	5	for	$l_i + r_i = 7$
given	**3**	**4**	**for**	$l_i + r_i = 7$
	4	3	for	$l_i + r_i = 7$
	5	2	for	$l_i + r_i = 7$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 10m\,,$$
$$P = 10m + 8\,.$$

In this example for both types, we obtain when $m = 1$ the following:

$$c = 2m - 1 = 1\,,$$
$$x = 10\,,$$
$$P = 18\,.$$

B^* should not have a divisor greater than 1 in common with 18 .

Therefore: **B^* is coprime with both 2 and 3 .**

Example 2:

Example 2(a) Type 2 Given $l_i = 3$; $r_i = 5$; $A = 5$.

$$\Delta_1 = 1\,,\text{ and so } \Delta_2 = A - 1 = 4\,.$$
$$\delta = |2(3+5) - 4|_5 = 2\,.$$

$$l_i + r_i = 3 + 5 = 8 = Y + A\,,$$

hence also $l_i + r_i = Y = 8 - 5 = 3$.

Therefore:

$$c = (2m - 3) + \frac{(2 \times 8) - 2 + 1}{5} = 2m \qquad (c = \text{even})\,,$$

and

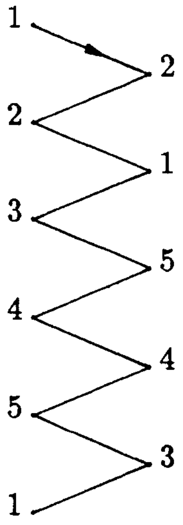
$$c = (2m^* - 3) + \frac{(2 \times 3) - 2 + 1}{5} = 2m^* - 2 \qquad (c = \text{even})\,.$$

Therefore:

$$2m = 2m^* - 2\,, \qquad \text{hence } m = m^* - 1\,.$$
$$x = cA + \delta = 10m + 2\,,$$
$$P_{total} = P = 10m + 10\,.$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 10m + 2\,,$$
$$P = 10m + 10\,.$$

Example 2(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (2(a)).

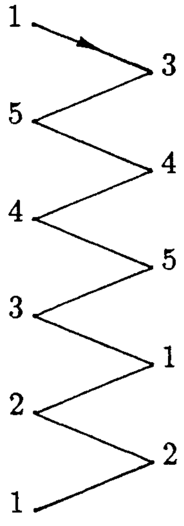
For $l_i = 3$, we obtain:

$$r_{i_2} = |r_{i_1} + \Delta_2|_A = |r_{i_1} + 4|_5 = 5\,, \text{ hence } \qquad r_{i_1} = r_i = 1\,.$$
$$l_i + r_i = 3 + 1 = 4\,,$$
$$Y = 4 \qquad \text{hence also } l_i + r_i = Y + A = 4 + 5 = 9\,.$$

As before, it is evident that $\delta_{type1} = \delta_{type2}$, and also that $c_{type1} = c_{type2}$ for mirror-imaged pairs.

	l_i	r_i		
	1	3	for	$l_i + r_i = 4$
	2	2	for	$l_i + r_i = 4$
given	**3**	**1**	**for**	$l_i + r_i = 4$
	4	5	for	$l_i + r_i = 9$
	5	4	for	$l_i + r_i = 9$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 10m + 2\,,$$
$$P = 10m + 10\,.$$

In this example for both types, we obtain when $m = 1$ the following:

$$c = 2m = 2\,,\quad x = 12\,,\quad P = 20\,.$$

B^* should not have a divisor greater than 1 in common with 20 .

Therefore: **B^* is coprime with both 2 and 5 .**

(4) — SEMI-PERFECT HERRINGBONE PINEAPPLE KNOTS

$$\begin{aligned} \Delta_1 &= 1 && \text{(Type 1)}\,, \\ \Delta_2 &= A - 1 && \text{(Type 2)}\,. \end{aligned}$$

B^* and P_{total} have a common divisor greater than 1.

The Semi-Perfect Herringbone Pineapple Knots are like the Perfect Herringbone Pineapple Knots, except that they require more than one string in their construction.

All formulae derivations are as in **(3)** above.

Example 1 :

Example 1(a) Type 1, and Example 1(b) Type 2

Using the same numerical values for the various parameters as in examples 1(a) and 1(b) above, under **(3)**, we find that the only difference for the Semi-Perfect Herringbone Pineapple Knot is that now B^* has a divisor of 2 and/or 3.

Example 2 :

Example 2(a) Type 2, and Example 2(b) Type 1

Using the same numerical values for the various parameters as in examples 2(a) and 2(b) above, under **(3)**, we find that the only difference for the Semi-Perfect Herringbone Pineapple Knot is that now B^* has a divisor of 2 and/or 5.

(5) — STANDARD BROKEN-HERRINGBONE PINEAPPLE KNOTS

$\Delta = 0$; B^* and $P_{component}$ are coprime.

$$P_{total} = P = \sum P_{component} = x + 2A - 2\,,$$

$$P_{component} = \frac{x + 4A - 2(l_i + r_i)}{A}\;; \qquad (P_{component} \text{ is even})\,,$$

$$x = cA + \delta\,,$$

$$c \neq (2m - 3) + \frac{2(l_i + r_i) - \delta}{A}\,.$$

Note: All components of a given knot do NOT necessarily have the same m-value (see the following examples).

Example :

Given $l_i = 3\ ;\ r_i = 5\ ;\ A = 5\ .$

Therefore :

$$\delta = |2(l_i + r_i) - \Delta|_A = |2(3+5) - 0|_5 = |16|_5 = 1\ .$$

$l_i + r_i = 3 + 5 = 8$ hence $Y = 3$ and hence also $l_i + r_i = 3$.

Therefore :

$$c \neq (2m - 3) + \frac{2(l_i + r_i) - \delta}{A}\ ,$$

$$c \neq (2m - 3) + \frac{(2 \times 8) - 1}{5}\ ,$$

$$c \neq 2m \qquad \text{hence} \quad c = \text{odd}\ ,$$

and

$$c \neq (2m^* - 3) + \frac{(2 \times 3) - 1}{5}\ ,$$

$$c \neq 2m^* - 2 \qquad \text{hence} \quad c = \text{odd}\ .$$

Therefore :

$$c = 2m - 1\ ,$$

$$x = cA + \delta = 10m - 4\ ,$$

$$P_{component} = \frac{10m - 4 + (4 \times 5) - (2 \times 8)}{5} = 2m \qquad \text{for} \quad l_i + r_i = 8\ ,$$

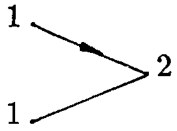
and

$$P_{component} = \frac{10m - 4 + (4 \times 5) - (2 \times 3)}{5} = 2m + 2 \qquad \text{for} \quad l_i + r_i = 3\ ,$$

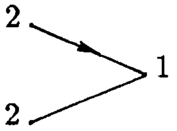
$$P = P_{total} = 10m - 4 + (2 \times 5) - 2 = 10m + 4\ .$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

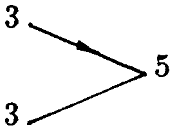
The first-return string-runs of the components are therefore as follows:



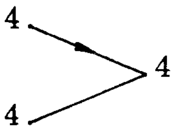
$$P_{comp} = 2m + 2\,, \qquad (x = 10m - 4)\,.$$



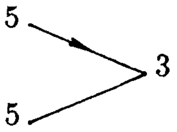
$$P_{comp} = 2m + 2\,, \qquad (x = 10m - 4)\,.$$



$$P_{comp} = 2m\,, \qquad (x = 10m - 4)\,.$$



$$P_{comp} = 2m\,, \qquad (x = 10m - 4)\,.$$



$$P_{comp} = 2m\,, \qquad (x = 10m - 4)\,.$$

For $m = 2$ we obtain:

$$2m + 2 = 6\,, \qquad \text{and}$$
$$2m = 4\,.$$
$$c = 2m - 1 = 3\,,$$
$$x = 10m - 4 = 16\,.$$
$$P_{component} = 6 \quad \text{and} \quad P_{component} = 4\,,$$
$$P_{total} = 24\,.$$

B^* is coprime to 2 and 3.

(6) — SEMI-STANDARD BROKEN-HERRINGBONE PINEAPPLE KNOTS

$\Delta = 0$; B^* and $P^*_{component}$ have a common divisor greater than 1.

$P^*_{component} = P_{component}$ as calculated in (**5**).

All derivations are as in (**5**).

See the Example in (**5**), where we obtained for $m = 2$:

$$2m = 4 \,, \qquad \text{and}$$
$$2m + 2 = 6 \,.$$
$$c = 2m - 1 = 3 \,,$$
$$x = 10m - 4 = 16 \,.$$
$$P_{component} = 6 \quad \text{and} \quad P_{component} = 4 \,.$$

Hence
$$P^*_{component} = 6 \quad \text{and} \quad P^*_{component} = 4 \,,$$
$$P_{total} = 24 \,.$$

B^* has a divisor of 2 and/or 3 .

(7) — PERFECT BROKEN-HERRINGBONE PINEAPPLE KNOTS

Δ can take any value in the range $1, \ldots, (A - 1)$ which is coprime with A .

B^* and $P = P_{total}$ are coprime.

$$\begin{array}{lll} \text{Type 1:} & \Delta_1 = \epsilon & 1 \le \epsilon < A/2 \,, \\ \text{Type 2:} & \Delta_2 = A - \epsilon & 1 \le \epsilon < A/2 \,. \end{array}$$

Thus: ϵ and A are coprime.

The Perfect Broken-Herringbone Pineapple Knot has one component only, and hence
$P_{component} = P_{total} = P = x + 2A - 2$.

For $\epsilon = 1$:
$$c \neq (2m - 3) + \frac{2(l_i + r_i) - \delta - 1}{A} \qquad \text{(Type 1)} \,,$$
$$c \neq (2m - 3) + \frac{2(l_i + r_i) - \delta + 1}{A} \qquad \text{(Type 2)} \,.$$

Example 1:

Example 1(a) Type 1 Given $\epsilon = 1$; $l_i = 3$; $r_i = 5$; $A = 5$.

$\epsilon = 1$ implies $\Delta_1 = 1$: $\delta = |2(3 + 5) - 1|_5 = 5$.

$l_i + r_i = 3 + 5 = 8$,

$Y + A = Y + 5 = 8$ therefore $Y = 3$ and hence also $l_i + r_i = 3$.

Therefore:

$$\begin{aligned} c &\neq (2m-3) + \frac{(2\times 8)-5-1}{5} \\ &\neq 2m-1 \qquad (c = \text{even}), \end{aligned}$$

and

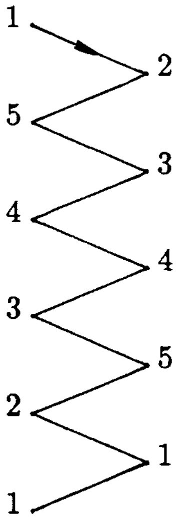
$$\begin{aligned} c &\neq (2m^*-3) + \frac{(2\times 3)-5-1}{5}, \\ &\neq 2m^*-3 \qquad (c = \text{even}). \end{aligned}$$

Therefore:

$$\begin{aligned} c &= 2m, \\ x &= cA + \delta = 10m+5, \\ P_{total} &= P = 10m+13. \end{aligned}$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$\begin{aligned} x &= 10m+5, \\ P &= 10m+13. \end{aligned}$$

Example 1(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (1(a)).

For $l_i = 3$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |5-1|_5 = 4.$$

$$l_i + r_i = 3 + 4 = 7,$$

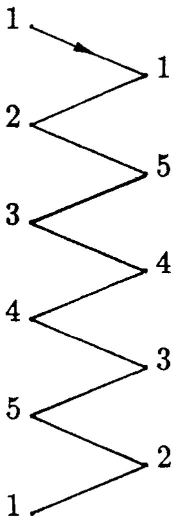
$Y + A = Y + 5 = 7$ therefore $Y = 2$ and hence also $l_i + r_i = 2$.

$$\Delta_2 = 5 - 1 = 4.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	1	1	for	$l_i + r_i = 2$
	2	5	for	$l_i + r_i = 7$
given	**3**	**4**	**for**	$l_i + r_i = 7$
	4	3	for	$l_i + r_i = 7$
	5	2	for	$l_i + r_i = 7$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 10m + 5 \, ,$$
$$P = 10m + 13 \, .$$

In this example for both Types, we obtain when $m = 1$ the following:

$$c = 2m = 2 \, ,$$
$$x = 15 \, ,$$
$$P = 23 \, .$$

Since B^* and P are coprime, B^* must be coprime with 23 .

Example 2 :

Example 2(a) Type 1 Given $\epsilon = 2$; $l_i = 3$; $r_i = 5$; $A = 5$.

$\epsilon = 2$ implies $\Delta_1 = 2$: $\delta = |2(3+5) - 2|_5 = 4$.

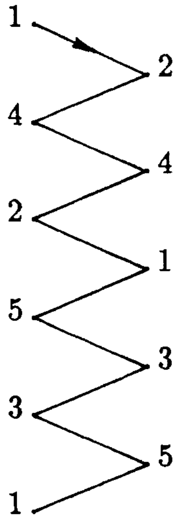
$l_i + r_i = 3 + 5 = 8$,

$Y + A = Y + 5 = 8$ therefore $Y = 3$ and hence also $l_i + r_i = 3$.

$$x = cA + \delta = 5c + 4 \, ,$$
$$P_{total} = P = x + 2A - 2 = 5c + 12 \, .$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 2$):



$x = 5c + 4\,,$
$P = 5c + 12\,.$

Example 2(b) Type 2

We now give details of the mirror-image of the foregoing Type 1 example (2(a)).

For $l_i = 3$, we obtain:

$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |5-2|_5 = 3\,.$$

$$l_i + r_i = 3 + 3 = 6 = A + 1\,,$$

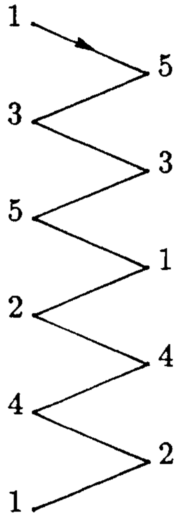
Hence for all $l_i + r_i$ we obtain the value 6.

$$\Delta_2 = 5 - 2 = 3\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	1	5	for	$l_i + r_i = 6$
	2	4	for	$l_i + r_i = 6$
given	**3**	**3**	**for**	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	5	1	for	$l_i + r_i = 6$

The first-return string-run is therefore as follows (with $\Delta = 3$):



$x = 5c + 4\,,$
$P = 5c + 12\,.$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 14\,,$$
$$P = 22\,.$$

Since B^* and P are coprime, B^* must be coprime with 2 and 11.

Example 3:

Example 3(a) Type 2 Given $\epsilon = 2$; $l_i = 3$; $r_i = 5$; $A = 5$.

$$\epsilon = 2 \text{ implies } \Delta_2 = A - \epsilon = 3 \ : \qquad \delta = |2(4+1)+3|_5 = 3\,.$$

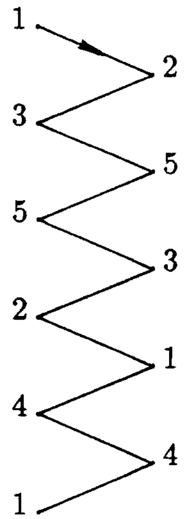
$$l_i + r_i = 3 + 5 = 8 = Y + A\,,$$

hence also $l_i + r_i = Y = 3$.

$$x = cA + \delta = 5c + 3\,,$$
$$P_{total} = P = 5c + 11\,.$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 3$):



$$x = 5c + 3\,,$$
$$P = 5c + 11\,.$$

Example 3(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (3(a)).

For $l_i = 3$, we obtain:

$$r_{i_2} = |r_{i_1} + \Delta_2|_A = |r_{i_1} + 3|_5 = 5\,, \quad \text{hence} \quad r_{i_1} = r_i = 2\,.$$

$$\Delta_1 = \epsilon = A - \Delta_2 = 5 - 3 = 2\,.$$

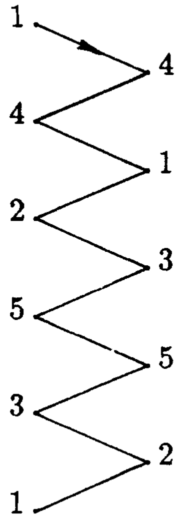
$$l_i + r_i = 3 + 2 = 5\,,$$

$$Y = 5 \qquad \text{hence also } l_i + r_i = Y + A = 5 + 5 = 10\,.$$

As before, it is evident that $\delta_{type1} = \delta_{type2}$, and also that $c_{type1} = c_{type2}$ for mirror-imaged pairs.

	l_i	r_i		
	1	4	for	$l_i + r_i = 5$
	2	3	for	$l_i + r_i = 5$
given	**3**	**2**	**for**	$l_i + r_i = 5$
	4	1	for	$l_i + r_i = 5$
	5	5	for	$l_i + r_i = 10$

The first-return string-run is therefore as follows (with $\Delta = 2$):



$$x = 5c + 3 \,,$$
$$P = 5c + 11 \,.$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 13 \,,$$
$$P = 21 \,.$$

B^* should not have a divisor greater than 1 in common with 21 ,

Therefore: **B^* is coprime with 3 and 7.**

Example 4:

Example 4(a) Type 2 Given $\epsilon = 1$; $l_i = 3$; $r_i = 5$; $A = 5$.

$\epsilon = 1$ implies $\Delta_2 = A - \epsilon = 4$: $\delta = |2(3+5) - 4|_5 = 2$.

$l_i + r_i = 3 + 5 = 8$,

$Y + A = Y + 5 = 8$ therefore $Y = 3$ and hence also $l_i + r_i = 3$.

Therefore:

$$c \neq (2m - 3) + \frac{(2 \times 8) - 2 + 1}{5}$$
$$\neq 2m \qquad (c = \text{odd}) \,,$$

and

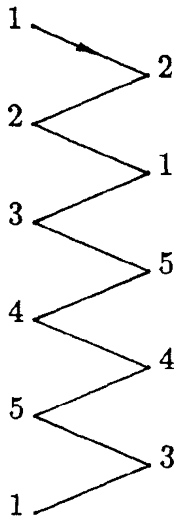
$$c \neq (2m^* - 3) + \frac{(2 \times 3) - 2 + 1}{5}$$
$$\neq 2m^* - 2 \qquad (c = \text{odd}) \,.$$

Therefore:

$$c = 2m - 1\,,$$
$$x = cA + \delta = 10m - 3\,,$$
$$P_{total} = P = 10m + 5\,.$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
	2	1	for	$l_i + r_i = 3$
given	**3**	**5**	**for**	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$

The first-return string-run is therefore as follows (with $\Delta = 4$):



$$x = 10m - 3\,,$$
$$P = 10m + 5\,.$$

Example 4(b) Type 1

We now give details of the mirror-image of the foregoing Type 2 example (4(a)).

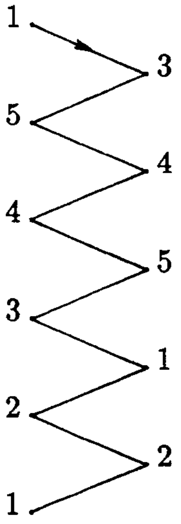
For $l_i = 3$, we obtain:

$$r_i = r_{i_1} = |r_{i_2} - \Delta_2|_A = |5 - 4|_5 = 1\,.$$
$$l_i + r_i = 3 + 1 = 4\,,$$
$$Y = 4 \qquad \text{hence also } l_i + r_i = Y + A = 4 + 5 = 9\,,$$
$$\Delta_1 = \epsilon = A - \Delta_2 = 1\,.$$

It is evident that $\delta_{type2} = \delta_{type1}$, and also that $c_{type2} = c_{type1}$ for mirror-imaged pairs.

	l_i	r_i		
	1	3	for	$l_i + r_i = 4$
	2	2	for	$l_i + r_i = 4$
given	**3**	**1**	**for**	$l_i + r_i = 4$
	4	5	for	$l_i + r_i = 9$
	5	4	for	$l_i + r_i = 9$

The first-return string-run is therefore as follows (with $\Delta = 1$):



$$x = 10m - 3\,,$$
$$P = 10m + 5\,.$$

In this example for both types, we obtain when $m = 1$ the following:

$$c = 2m - 1 = 1\,,$$
$$x = 7\,,$$
$$P = 15\,.$$

Since B^* and P are coprime, B^* must be coprime with 3 and 5.

(8) — SEMI-PERFECT BROKEN-HERRINGBONE PINEAPPLE KNOTS

Δ can take any value in the range $1, \ldots, (A-1)$ which is coprime with A.

B^* and $P = P_{total}$ have a common divisor greater than 1.

The Semi-Perfect Broken-Herringbone Pineapple Knots are like the Perfect Broken-Herringbone Pineapple Knots, except that they require more than one string in their construction.

All formulae derivations are as in **(7)** above.

Example 1:

Example 1(a) Type 1, and Example 1(b) Type 2

Using the same numerical values for the various parameters as in examples 1(a) and 1(b) under **(7)**, we find that the only difference for the Semi-Perfect Broken-Herringbone Pineapple Knot is that now B^* and P must have a common divisor of 23.

Example 2:

Example 2(a) Type 1, and Example 2(b) Type 2

Using the same numerical values for the various parameters as in examples 2(a) and 2(b) above, under **(7)**, we find that the only difference for the Semi-Perfect Broken-Herringbone Pineapple Knot is that now B^* and P must have a common divisor of 2 and/or 11.

Example 3 :

Example 3(a) Type 2, and Example 3(b) Type 1

Using the same numerical values for the various parameters as in examples 3(a) and 3(b) under **(7)**, we find that the only difference for the Semi-Perfect Broken-Herringbone Pineapple Knot is that now B^* and P must have a common divisor of 3 and/or 7 .

Example 4 :

Example 4(a) Type 2, and Example 4(b) Type 1

Using the same numerical values for the various parameters as in examples 4(a) and 4(b) under **(7)**, we find that the only difference for the Semi-Perfect Broken-Herringbone Pineapple Knot is that now B^* and P must have a common divisor of 3 and/or 5 .

(9) — COMPOUND BROKEN-HERRINGBONE PINEAPPLE KNOTS

For knots in this sub-class, Δ can take any value in the range $2, \ldots, (A-2)$ which has with A a common divisor.

$$B^* \text{ and } P_{component} \text{ are coprime.}$$

The number of passes is the same for each component of a given knot.

We define the number of passes per component by α. Hence the number of components of a given knot is A/α .

When a component is completed, α bight-boundaries on each edge of the knot have been visited, and then the equation $|1 + \alpha \cdot \Delta|_A = 1$ is satisfied. This implies that $\alpha \cdot \Delta$ is an integral multiple (say σ) of A ; that is $\alpha \cdot \Delta = \sigma \cdot A$; and so, finally, we can write:

$$\Delta = \frac{\sigma \cdot A}{\alpha} .$$

We may note that σ and α are coprime.

$$\text{Type 1 :} \quad \Delta_1 = \epsilon \qquad 2 \le \epsilon < A/2 ,$$
$$\text{Type 2 :} \quad \Delta_2 = A - \epsilon \qquad 2 \le \epsilon < A/2 .$$

Note that Type 1 and Type 2 have the same Δ-value when $\Delta_1 = \Delta_2 = A/2$.

Furthermore, since Δ and A have a common divisor greater than 1 , ϵ and A are not coprime.

$$x = cA + \delta \qquad \text{where} \quad c = (0), 1, 2, 3, \ldots ,$$
$$P_{component} = P_\alpha = \alpha(c+4) + \left\{ \frac{\alpha \cdot \delta - 2(\Sigma l_i + \Sigma r_i)}{A} \right\} ,$$
$$P_{total} = \Sigma P_{component} = x + 2A - 2 .$$

Example 1 :

Given $l_i = 2$; $r_i = 4$; $A = 6$.

$A = 6$ has the divisors 2 and 3 ; thus $\alpha = 2$ or 3.

Let α be equal to 2 ; then $\Delta = \sigma A/2 = 3\sigma$.

But since $2 \le \Delta \le A - 2$, we obtain $\sigma = 1$ and hence $\Delta = 3$.

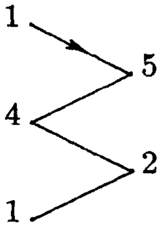
Example 1(a)

$$\delta = |2(2+4) - 3|_6 = 3\,,$$
$$x = cA + \delta = 6c + 3\,,$$
$$P = P_{total} = x + 2A - 2 = 6c + 13\,.$$

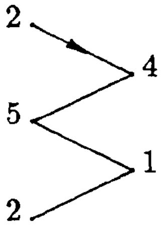
$$l_i + r_i = 2 + 4 = 6\,,$$
$$Y = 6 \qquad \text{hence also } l_i + r_i = Y + A = 6 + 6 = 12\,.$$

	l_i	r_i		
	1	5	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	5	1	for	$l_i + r_i = 6$
	6	6	for	$l_i + r_i = 12$

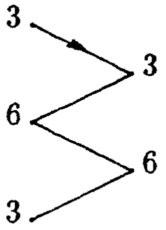
The first-return string-runs are therefore as follows (with $\Delta = 3$):



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(1+4+5+2)}{6} = 2c + 5\,.$$



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(2+5+4+1)}{6} = 2c + 5\,.$$



$$P_{component} = 2(c+4) + \frac{(2 \times 3) - 2(3+6+3+6)}{6} = 2c + 3\,.$$

Example 1(b)

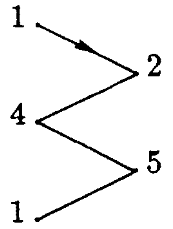
We now give details of the mirror-image of the foregoing example (1(a)).

For $l_i = 2$, we obtain:

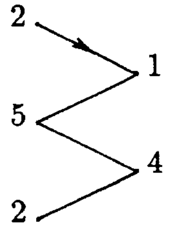
$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |4 - 3|_6 = 1\,.$$
$$l_i + r_i = 2 + 1 = 3\,,$$
$$Y = 3 \qquad \text{hence also } l_i + r_i = Y + A = 3 + 6 = 9\,.$$
$$\Delta_2 = A - \Delta_1 = 6 - 3 = 3 = \Delta_1\,.$$

	l_i	r_i		
	1	2	for	$l_i + r_i = 3$
given	**2**	**1**	**for**	$l_i + r_i = 3$
	3	6	for	$l_i + r_i = 9$
	4	5	for	$l_i + r_i = 9$
	5	4	for	$l_i + r_i = 9$
	6	3	for	$l_i + r_i = 9$

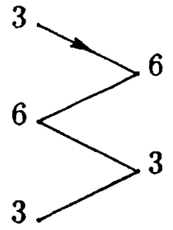
The first-return string-runs are therefore as follows (with $\Delta = 3$):



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(1+4+2+5)}{6} = 2c+5\ .$$



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(2+5+1+4)}{6} = 2c+5\ .$$



$$P_{component} = 2(c+4) + \frac{(2\times 3) - 2(3+6+6+3)}{6} = 2c+3\ .$$

It is of course inconsequential which of the above two knots 1(a) or 1(b) is called Type 1 or Type 2. We shall however indicate the knots which have a half-cycle $^{1}\searrow_{1}$; $^{1}\searrow_{2}$ and $^{1}\searrow_{3}$ as Type 1 knots and hence the knots which have a half-cycle $^{1}\searrow_{4}$; $^{1}\searrow_{5}$ and $^{1}\searrow_{6}$ as Type 2 knots.

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 3 = 15\ ,$$
$$P_{total} = P = 6c + 13 = 25\ ,$$
$$P_{component} = 9 \text{ and } P_{component} = 7\ .$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 3 and 7.

Example 2 :

Given $l_i = 2$; $r_i = 4$; $A = 6$.

Let α be equal to 3 ; then $\Delta = \dfrac{\sigma A}{3} = 2\sigma$.

Since $2 \leq \Delta \leq A - 2$, hence $2 \leq \Delta \leq 4$, the value of σ can be 1 or 2.

For $\sigma = 1$ we obtain $\Delta = 2$, and for $\sigma = 2$ we obtain $\Delta = 4$.

Example 2(a) Type 1

$$\Delta = 2 .$$

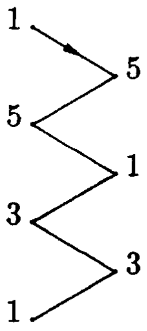
$$\delta = |2(2+4) - 2|_6 = 4 ,$$
$$x = cA + \delta = 6c + 4 ,$$
$$P = P_{total} = x + 2A - 2 = 6c + 14 .$$

$l_i + r_i = 2 + 4 = 6$,

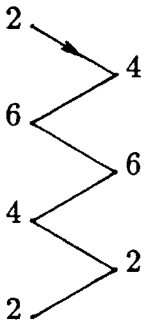
$Y = 6$ hence also $l_i + r_i = Y + A = 6 + 6 = 12$.

	l_i	r_i		
	1	5	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	5	1	for	$l_i + r_i = 6$
	6	6	for	$l_i + r_i = 12$

The first-return string-runs are therefore as follows (with $\Delta = 2$):



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(1+5+3+5+1+3)}{6} = 3c + 8 \cdot$$



$$P_{component} = 3(c+4) + \frac{(3 \times 4) - 2(2+6+4+4+6+2)}{6} = 3c + 6 \cdot$$

Example 2(b) Type 2

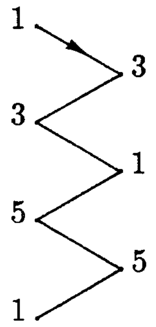
We now give details of the mirror-image of the foregoing Type 1 example (2(a)).

For $l_i = 2$, we obtain:

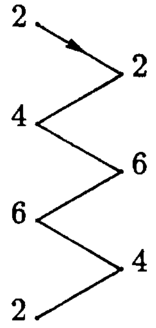
$$r_i = r_{i_2} = |r_{i_1} - \Delta_1|_A = |4-2|_6 = 2\,.$$
$$l_i + r_i = 2 + 2 = 4\,,$$
$$Y = 4 \qquad \text{hence also } l_i + r_i = Y + A = 4 + 6 = 10\,.$$
$$\Delta_2 = A - \Delta_1 = 6 - 2 = 4\,.$$

	l_i	r_i		
	1	3	for	$l_i + r_i = 4$
given	**2**	**2**	**for**	$l_i + r_i = 4$
	3	1	for	$l_i + r_i = 4$
	4	6	for	$l_i + r_i = 10$
	5	5	for	$l_i + r_i = 10$
	6	4	for	$l_i + r_i = 10$

The first-return string-runs are therefore as follows (with $\Delta = 4$):



$$P_{component} = 3(c+4) + \frac{(3\times 4) - 2(1+3+5+3+1+5)}{6} = 3c + 8\,\cdot$$



$$P_{component} = 3(c+4) + \frac{(3\times 4) - 2(2+4+6+2+6+4)}{6} = 3c + 6\,\cdot$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 4 = 16\,,$$
$$P_{total} = P = 6c + 14 = 26\,,$$
$$P_{component} = 14 \text{ and } P_{component} = 12\,.$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 2, 3 and 7.

Example 3 :

Given $l_i = 2\,;\ r_i = 4\,;\ A = 6\,.$

Let α be equal to 3; then $\Delta = \dfrac{\sigma A}{3} = 2\sigma\,.$

Since $2 \leq \Delta \leq A - 2$, hence $2 \leq \Delta \leq 4$, the value of σ can be 1 or 2.

For $\sigma = 1$ we obtain $\Delta = 2$, and for $\sigma = 2$ we obtain $\Delta = 4$.

Example 3(a) Type 2

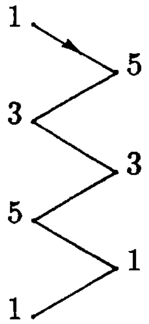
$$\Delta = 4\,.$$

Note that we are dealing with a Type 2 since $\Delta = 4$ and hence $\epsilon = 2$ (Type 2: $\Delta_2 = A - \epsilon$ with $2 \leq \epsilon < A/2$).

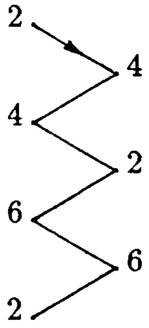
$$\delta = |2(2+4) - 4|_6 = 2\,,$$
$$x = cA + \delta = 6c + 2\,,$$
$$P = P_{total} = x + 2A - 2 = 6c + 12\,.$$
$$l_i + r_i = 2 + 4 = 6\,,$$
$$Y = 6 \qquad \text{hence also } l_i + r_i = Y + A = 6 + 6 = 12\,.$$

	l_i	r_i		
	1	5	for	$l_i + r_i = 6$
given	**2**	**4**	**for**	$l_i + r_i = 6$
	3	3	for	$l_i + r_i = 6$
	4	2	for	$l_i + r_i = 6$
	5	1	for	$l_i + r_i = 6$
	6	6	for	$l_i + r_i = 12$

The first-return string-runs are therefore as follows (with $\Delta = 4$):



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(1+3+5+5+3+1)}{6} = 3c + 7\,.$$



$$P_{component} = 3(c+4) + \frac{(3 \times 2) - 2(2+4+6+4+2+6)}{6} = 3c + 5\,.$$

Example 3(b) Type 1

We now give details of the mirror-image of the foregoing example (3(a) Type 2).

For $l_i = 2$, we obtain:

$$r_i = r_{i_1} = |r_{i_2} - \Delta_2|_A = |4-4|_6 = 6\,.$$

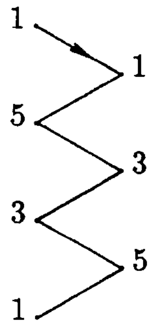
$$l_i + r_i = 2 + 6 = 8\,,$$

$$Y + A = 8 \qquad \text{therefore } Y = 2 \text{ and hence also } l_i + r_i = 2\,.$$

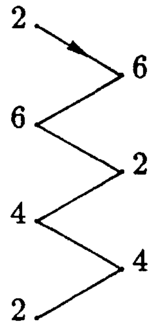
$$\Delta_1 = A - \Delta_2 = 6 - 4 = 2\,.$$

	l_i	r_i		
	1	1	for	$l_i + r_i = 2$
given	**2**	**6**	**for**	$l_i + r_i = 8$
	3	5	for	$l_i + r_i = 8$
	4	4	for	$l_i + r_i = 8$
	5	3	for	$l_i + r_i = 8$
	6	2	for	$l_i + r_i = 8$

The first-return string-runs are therefore as follows (with $\Delta = 2$):



$$P_{component} = 3(c+4) + \frac{(3\times 2) - 2(1+5+3+1+3+5)}{6} = 3c + 7\,.$$



$$P_{component} = 3(c+4) + \frac{(3\times 2) - 2(2+6+4+6+2+4)}{6} = 3c + 5\,.$$

In this example for both Types, we obtain when $c = 2$ the following:

$$x = 6c + 2 = 14\,,$$

$$P_{total} = P = 6c + 12 = 24\,,$$

$$P_{component} = 13 \text{ and } P_{component} = 11\,.$$

Since B^* and $P_{component}$ are coprime, B^* must be coprime with 13 and 11.

(10) — SEMI-COMPOUND BROKEN-HERRINGBONE PINEAPPLE KNOTS

For knots in this sub-class, Δ can take any value in the range $2, \ldots, (A-2)$ which has with A a common divisor.

B^* and $P^*_{component}$ have a common divisor greater than 1.

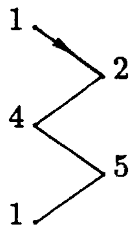
$P^*_{component} = P_{component}$ as calculated in **(9)**.

All derivations are as in **(9)**.

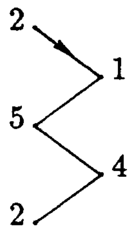
See the Examples in **(9)**:

Example 1:

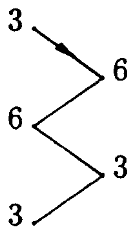
Example 1(a) and 1(b)



$P^*_{comp} =$
$2c+5$

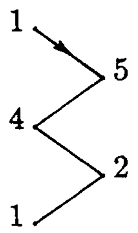


$P^*_{comp} =$
$2c+5$

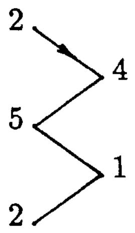


$P^*_{comp} =$
$2c+3$

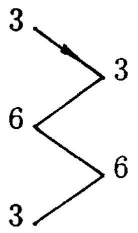
and



$P^*_{comp} =$
$2c+5$



$P^*_{comp} =$
$2c+5$



$P^*_{comp} =$
$2c+3$

For $c = 2$ we obtain:

$$x = 6c + 3 = 15,$$

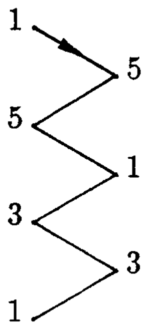
$$P_{total} = P = 6c + 13 = 25,$$

$$P^*_{component} = 9 \text{ and } P^*_{component} = 7.$$

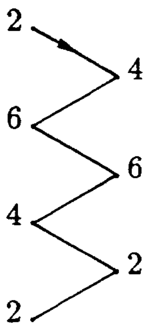
B^* and $P^*_{component}$ have a common divisor of 3 and/or 7.

Example 2:

Example 2(a) and 2(b)

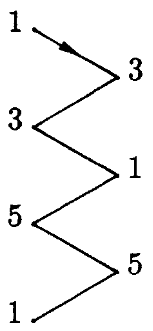


$P^*_{comp} =$
$3c+8$

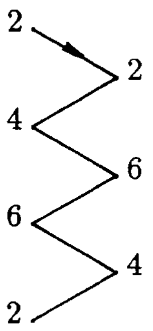


$P^*_{comp} =$
$3c+6$

and



$P^*_{comp} =$
$3c+8$



$P^*_{comp} =$
$3c+6$

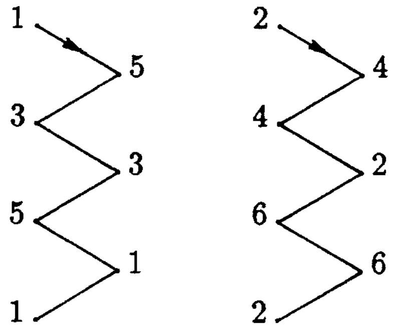
For $c = 2$ we obtain:

$$x = 6c + 4 = 16\,,$$
$$P_{total} = P = 6c + 14 = 26\,,$$
$$P^*_{component} = 14 \text{ and } P^*_{component} = 12\,.$$

B^* and $P^*_{component}$ have a common divisor of 2 and/or 3 and/or 7.

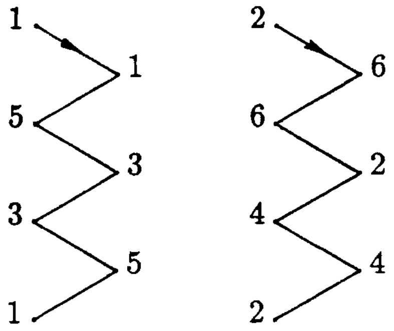
Example 3:

Example 3(a) and 3(b)



$P^*_{comp} = 3c + 7$ $\quad$ $P^*_{comp} = 3c + 5$

and



$P^*_{comp} = 3c + 7$ $\quad$ $P^*_{comp} = 3c + 5$

For $c = 2$ we obtain:

$$x = 6c + 2 = 14\,,$$
$$P_{total} = P = 6c + 12 = 24\,,$$
$$P^*_{component} = 13 \text{ and } P^*_{component} = 11\,.$$

B^* and $P^*_{component}$ have a common divisor of 11 and/or 13.

Appendix 3

MODULAR ARITHMETIC

When constructing cylindrical braids, a string is passed round and around a cylinder, forming half-cycles (from left-hand bights to right-hand bights and vice-versa) in a string-run as it goes. Each time it passes round the cylinder, the newly formed half-cycles may overlay those which were formed previously; and since no bight position is ever occupied twice, the half-cycles generally cross one another.

It is natural then, that in order to study relative bight and string crossing positions on the cylinder, we have to use what is called modular arithmetic (or clock arithmetic) to do our calculations.

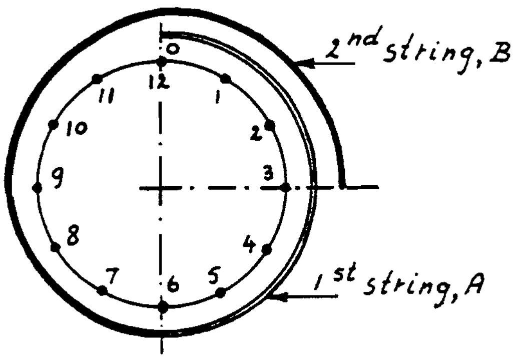
Suppose we mark twelve equi-distant points around a section of a cylinder (to use a '12-hour clock' example), and wrap two portions of string (say A and B) round it, in a clockwise direction.

Let the length of portion A be equal to 6 units, and the length of portion B be equal to 9 units.

The strings are joined end-to-end, so their total length is 15 units.

The string begins at position 0 (equivalent to position 12) and ends at position 3.

Thus, by 'ordinary' arithmetic, $A + B = 15$; whereas by 'clock arithmetic', we can write $A + B = 3$.



In effect, every time we pass completely round the cylinder we discard 12 units from our sum, and start counting from 0 again. This is called modulo-12 addition.

Our notation for this is $|A + B|_{12}$. We use the notation throughout all our books and research papers on Braiding.

The symbols indicate that the ordinary sum of the lengths A and B has to be reduced by as many multiples of 12 as it contains; the 'answer' is the smallest possible positive or zero remainder. In general, we refer to this value simply as the remainder.

Sometimes in our braiding applications, when it happens that the remainder is $r = 0$, we find it necessary to use the modulus-value (i.e. 12) instead of 0 (refer to the clock diagram, where it is seen that 0 and 12 are equivalent, and therefore interchangeable).

In everyday life, for example, we never say that the time is 0 o'clock, but always 12 o'clock.

Examples :

$$|3+4|_{12} = |7|_{12} = 7\,.$$
$$|9+10|_{12} = |19|_{12} = 7\,.$$
$$|11+5|_{12} = |16|_{12} = 4\,.$$
$$|6+6|_{12} = |12|_{12} = 0 \text{ (sometimes written as 12)}\,.$$

Note that we can find the modulus of a negative integer, and interpret this as wrapping the string around the cylinder in the anti-clockwise direction. For example, wrapping the strings $A+B=15$ anti-clockwise on the modulo-12 clock takes us to the point 9 on the clock. Hence we can write:

$$|-15|_{12} = |-3|_{12} = 9\,.$$

Further examples are:

$$|5-9|_{12} = |-4|_{12} = 8\,.$$
$$|6-11|_{12} = |-5|_{12} = 7\,.$$
$$|3-22|_{12} = |-19|_{12} = |-7|_{12} = 5\,.$$
$$|7+2-10|_{12} = |-1|_{12} = 11\,.$$

Numbers other than 12 may be used as a modulus. Once the modulus is given, a complete system of addition, subtraction, multiplication and division may be defined, in a similar fashion to that of ordinary arithmetic.

Examples :

$$|9|_3 = 0 \text{ (sometimes written as 3)}\,.$$
$$|21|_5 = 1\,.$$
$$|8+15|_6 = |23|_6 = |5|_6 = 5\,.$$
$$|-7|_4 = |-3|_4 = 1\,.$$
$$\big||8|_6 + |15|_6\big|_4 = |2+3|_4 = |5|_4 = |1|_4 = 1\,.$$

In general notation, suppose the modulus is p. Then if N is any integer, it can be expressed in the form:

$$N = m \cdot p + r\,,$$

where m is the number of multiples of p which causes the remainder r to be a number in the interval $0 \le r < p$.

Note that m may be any integer; that is, m may be any whole number, positive, negative or zero.

Our general notation for the modulus operation is:

$$|N|_p = r\,.$$

Below we give some Rules associated with modular arithmetic. The reader may care to make up examples and demonstrate the truth of each one.

Rules:

(1) If $|A|_p = |B|_p$, and $|C|_p = |D|_p$, then:
$|A + C|_p = |B + D|_p$,
$|A - C|_p = |B - D|_p$,
$|A \cdot C|_p = |B \cdot D|_p$.

(2) $|A + B|_p = \Big||A|_p + |B|_p\Big|_p$.

(3) $|n \cdot A|_{n \cdot p} = n \cdot |A|_p$.

(4) If $|A|_p = |B|_p$, we may **not** deduce that $A = B$;
we can, however, deduce that $|A - B|_p = 0$.
Conversely, $|A - B|_p = 0$ implies that $|A|_p = |B|_p$.

(5) If $|k \cdot A|_p = |k \cdot B|_p$, then:
$|A|_{\frac{p}{d}} = |B|_{\frac{p}{d}}$, where g.c.d.$(k, p) = d$.

In number theory the following notation, originating from the great German mathematician K.F.Gauss (1777 - 1855), is much used:

$A \equiv B$ (modulo p) means that $A - B$ is divisible by p. It is to be read thus:
A is congruent to B, mod p.

From (4) above, we see that '$A \equiv B$ (modulo p)' is equivalent to '$|A|_p = |B|_p$'. Hence congruence arithmetic and modular arithmetic are the same thing.

We have found our notation more convenient than Gauss's, for developing our theories of braiding.

Appendix 4

RELATIONSHIPS BETWEEN BRAIDING ALGORITHM CALCULATIONS

In Fig. 64 of Chapter 7 we gave a table and discussed its function in relation to the calculation of the Algorithm-tables which supplied us with the braiding algorithms for the Standard Herringbone Pineapple Knots. Recall that the table in Fig. 64 gave us the **set-numbers** where intersections **additional to** the reference quantities occurred. By comparing this table with the one given below in Fig. 65 the reader will observe their striking similarities. The table in Fig. 65 can be used in calculating the braiding algorithms for the Regular Knots in general. Although the calculation of braiding algorithms for Regular Knots as described in Book 1/1: *BRAIDING — Regular Knots* may seem at first sight not to have much in common with the table in Fig. 65; however the methods are in fact the same. In the Fig. 65 table, b stands for the number of bights of the Regular Knot and p stands for its number of parts.

The table works as follows:

The 1^{st} half-cycle is always a free run, hence the table gives for the 1^{st} half-cycle '**NONE**', indicating that there are no columns where intersections occur.

The 2^{nd} half-cycle gives mb, where m can have the values $1, 2, 3, \cdots$. Column-numbers outside the range $1, 2, 3, \cdots, (p-1)$ have to be discarded. Suppose we want to braid a Regular Knot having 15 parts and 7 bights; then the valid values for mb are 7 and 14 (1×7 and 2×7 respectively). Hence for half-cycle 2, intersections occur on the columns 7 and 14 as numbered from right to left, since half-cycle 2 runs from lower-right to upper left.

An **odd** half-cycle has the intersections on columns, whose **numbers**, when numbered from left to right (since an odd half-cycle runs from lower-left to upper-right), are the same as those of the columns which are intersected during the **immediately preceding even** half-cycle.

Calculation of the *COLUMN-NUMBERS*
where an intersection occurs:

Half-cycle	column-numbers
1.	NONE
2.	mb .
3.	" .
4.	" ; mb - p .
5.	" ; " .
6.	" ; " ; mb - 2p .
7.	" ; " ; " .
8.	" ; " ; " ; mb - 3p .
9.	" ; " ; " ; " .
10.	" ; " ; " ; " ; mb - 4p .
11.	" ; " ; " ; " ; " .
12.	" ; " ; " ; " ; " ; mb - 5p .
13.	" ; " ; " ; " ; " ; " .
14.	" ; " ; " ; " ; " ; " ; mb - 6p .
15.	" ; " ; " ; " ; " ; " ; " .
etc.	***etc.***
last half-cycle is **2b**	***where*** **m = 1,2,3, . . . ,** ***discard*** **column-numbers *outside the range* 1,2,3, . . . ,(p - 1) .**
2n	mb - rp , *where* m = 1,2,3, . . . *and* r = 0,1,2, . . . ,(n - 1) .
2n + 1	***as for half-cycle*** **2n**
where n = 1,2, . .	***discard column-numbers outside the range*** **1,2,3, . . . ,(p - 1).**

Fig. 65 — Calculation-table for intersection column-numbers (for Regular Knots).

Thus in our example, intersections occur on the columns 7 and 14 for half-cycle 3, with the columns being numbered from left to right.

The 4^{th} half-cycle intersects not only the columns whose numbers are the same as those of the 2^{nd} half-cycle (hence the columns 7 and 14, numbered from right to left), but in addition it intersects the columns whose numbers (when numbered from right to left) are indicated by $(mb - s)$, hence the **additional** columns whose numbers are 6 and 13 ($(3 \times 7) - 15$ and $(4 \times 7) - 15$ respectively). Thus the 4^{th} half-cycle intersects the columns $6, 7, 13$ and 14 (numbered from right to left).

The 5^{th} half-cycle intersects the columns $6, 7, 13$ and 14 (numbered from left to right).

The 6^{th} half-cycle intersects not only the columns whose numbers are the same as those of the 4^{th} half-cycle (hence the columns $6, 7, 13$ and 14, numbered from right to left), but in addition it intersects the columns whose numbers (when numbered from right to left) are indicated by $(mb - 2s)$, hence the **additional** columns whose numbers are 5 and 12 ($(5 \times 7) - (2 \times 15)$ and $(6 \times 7) - (2 \times 15)$ respectively). Thus the 6^{th} half-cycle intersects the columns $5, 6, 7, 12, 13$ and 14 (numbered from right to left).

And so on.

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